DENDRITIC CRYSTAL GROWTH

10

`Draft chapter last generated 2024-12-02; P.H. Trinh`

In section 1.1, we initiated our interest in exponential asymptotics via the presentation of the geometric model of crystal growth; the model seeks to describe the prototypical interfacial growth seen in fig. 10.1.

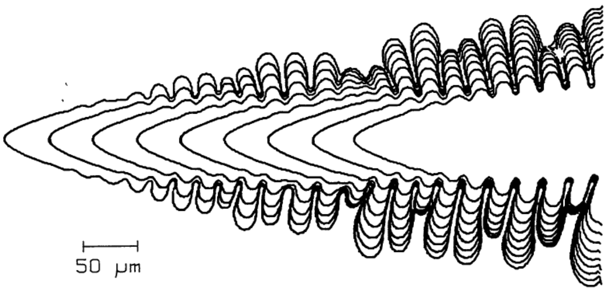

Figure 10.1: Contours of dendrite growth from a solution. Image from Dougherty et al. [1987].

Thus, following the proposal by Brower et al. [1983, 1984], we make the simplifying assumption that the crystal dynamics are governed purely by the local geometry of the interface. With $\phi = \phi(s)$ denoting the angle of the normal (with the respect to the horizontal, x-axis) of the interface, measured with arclength s, the following equation is developed (section 1.1):

$$\epsilon^2 \frac{\mathrm{d}^3\phi}{\mathrm{d}s^3} + \frac{\mathrm{d}\phi}{\mathrm{d}s} = \cos\phi, \tag{10.1a}$$

$$\phi \to \pm\frac{\pi}{2} \quad \text{as } s \to \pm\infty. \tag{10.1b}$$

Above, the assumption has been applied that the normal velocity of the interface is $v_n = \kappa + \epsilon^2 \kappa_{ss}$ where $\kappa = \frac{\mathrm{d}\phi}{\mathrm{d}s}$ is the surface curvature.[1]

[1]For further history, see Brower et al. [1983, 1984], Segur [1991], Gollub [1991], Dougherty et al. [1987].

Sufficiency of the boundary conditions

We can inspect the sufficiency of the two boundary conditions (10.1b) for the third-order problem. Near $s = -\infty$, let $\phi = -\pi/2 + f$, where $|f| \ll 1$. We then have

$$\epsilon^2 f''' + f' = f, \tag{10.2}$$

which must satisfy $f \to 0$ as $s \to -\infty$. Writing the solution as a sum of exponentials, with $f \sim \mathrm{e}^{ms}$, we find that[a]

$$f \sim a\mathrm{e}^{s} + \mathrm{e}^{-s/2}(b\mathrm{e}^{\mathrm{i}s/\epsilon} + c\mathrm{e}^{-\mathrm{i}s/\epsilon}). \tag{10.3}$$

Thus as $s \to -\infty$, the solution is composed of a permissible exponential decay, and also two oscillatory (growing) modes. The boundary condition at $s \to -\infty$ thus imposes the requirement that $b = 0 = c$ and is in fact, equivalent, to two boundary conditions. Similarly a linearisation about $s = \infty$ yields two additional boundary conditions. Including the translational invariance of the problem and thus the freedom to specify the origin in s, this yields a total of five boundary conditions on a third-order problem. The existence of a solution to the problem (10.1) would be unlikely "unless some miracle occurs" [Kruskal and Segur, 1991]. And yet this revelation is somewhat paradoxical to the ease in which we can derive sensible asymptotic solutions to all orders that satisfy the necessary boundary conditions.

[a] Exercise.

10.1 ESTABLISHING THE LATE ORDERS

We begin as usual with the regular perturbative expansion,

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^n \phi_n. \tag{10.4}$$

Noting that

$$\cos\phi = \cos\phi_0 - \epsilon^2 \phi_1 \sin\phi_0 + \mathcal{O}(\epsilon^4), \tag{10.5}$$

we have from (10.1a),

$$\mathcal{O}(1): \qquad \phi_0' = \cos\phi_0, \tag{10.6}$$

$$\mathcal{O}(\epsilon^2): \qquad \phi_1' + \phi_1 \sin\phi_0 = -\phi_0''', \tag{10.7}$$

$$\mathcal{O}(\epsilon^{2n}): \qquad \phi_n' + \phi_n \sin\phi_0 = -\phi_{n-1}''' + \ldots \tag{10.8}$$

At leading order, we have

$$\phi_0 = -\frac{\pi}{2} + 2\arctan \mathrm{e}^s. \tag{10.9}$$

Note that $\arctan u = \frac{\mathrm{i}}{2}\log\left(\frac{u+\mathrm{i}}{u-\mathrm{i}}\right)$

Note that this solution satisfies the necessary boundary conditions of $\phi_0 \to \pm\pi/2$ as $s \to \pm\infty$. Crucially, we note that the leading-order solution is singular at those singularities of $\arctan \mathrm{e}^s$ function, which occur at

$$s = \sigma_k = (2k+1)\pi\mathrm{i}/2 \tag{10.10}$$

for $k \in \mathbb{Z}$. The two closest singularities to the physical free surface (the real axis) are at $s = \sigma_\pm = \pm\pi\mathrm{i}/2$.

At next order, we have

$$\phi_1 = -(2 + s - 2\tanh s + C)\operatorname{sech} s, \tag{10.11}$$

where the constant of integration, C, is left undetermined at this stage. Notice that $\phi_1 \to 0$ as $s \to \pm\infty$ and hence the perturbative solution at this stage generically satisfies the boundary condition regardless of the choice of C (which fixes the origin). It can furthermore be shown [Kruskal and Segur, 1991] that a solution can be determined at every order that satisfies the boundary conditions.

As we have noted, the leading-order solution (10.9) has logarithmic singularities at $s = \sigma_k = (2k+1)\pi\mathrm{i}/2$ for $k \in \mathbb{Z}$. At next order, ϕ_1 depends on ϕ_0''' and hence we expect near $s = \sigma_k$,

$$\phi_1 \sim c(s - \sigma_k)^{-2}, \tag{10.12}$$

for some constant c. This pattern continues, with each subsequent order increasing the effect of the singularity from the previous order. In the limit $n \to \infty$, we expect factorial-over-power divergence of the form

$$\phi_n \sim \frac{Q(s)\Gamma(2n+\gamma)}{[\chi(s)]^{2n+\gamma}}. \tag{10.13}$$

Substitution into (10.8) gives

$$\left[-\frac{(\chi')^3 Q\Gamma(2n+\gamma+1)}{\chi^{2n+\gamma+1}} + \frac{\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}\left\{3\chi'\chi''Q + 3(\chi')^2 Q'\right\}\right]$$
$$+\left[-\frac{\chi' Q\Gamma(2n+\gamma+1)}{\chi^{2n+\gamma+1}} + \frac{Q'\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}\right] = \sin\phi_0 \frac{Q\Gamma(2n+\gamma)}{\chi^{2n+\gamma}} + \ldots \tag{10.14}$$

We divide by $\Gamma(2n+\gamma+1)$ and use the fact that

$$\frac{\Gamma(2n+\gamma)}{\Gamma(2n+\gamma+1)} \sim \frac{1}{2n} + \mathcal{O}(1/n^2). \tag{10.15}$$

Then at leading order as $n \to \infty$, we obtain an equation for the singulant function:

$$(\chi')^3 + \chi' = 0. \tag{10.16}$$

Thus the singulant functions that correspond to the singularity-generating divergences at $s = \sigma_k$ are given by

$$\chi(s) = \pm\mathrm{i}(s - \sigma_k), \tag{10.17}$$

where σ_k are those singularities up and down the imaginary axis given by (10.10).

Stokes lines of the crystal growth problem

As noted in the Universality Rules (chapter 9), Stokes lines are given by locations $s \in \mathbb{C}$ where $\operatorname{Im}\chi = 0$ and $\operatorname{Re}\chi \geq 0$. Focusing on the nearest singularities, we see that for $\sigma_+ = \pi\mathrm{i}/2$, considering the singulant with $\chi_+ = \mathrm{i}(s - \sigma_+)$ results in a Stokes line that follows the imaginary axis and intersects the origin, $s = 0$.

Similarly, the singularity $\sigma_- = -\pi\mathrm{i}/2$ has corresponding Stokes line with $\chi_- = -\mathrm{i}(s - \sigma_-)$ tending upwards along the imaginary axis. The two discussed Stokes lines are sketched in fig. 10.2.

At next order, we obtain an equation for Q, which yields

$$\frac{Q'}{Q} = \frac{1}{2}\sin\phi_0 = \frac{1}{2}\tanh s, \tag{10.18}$$

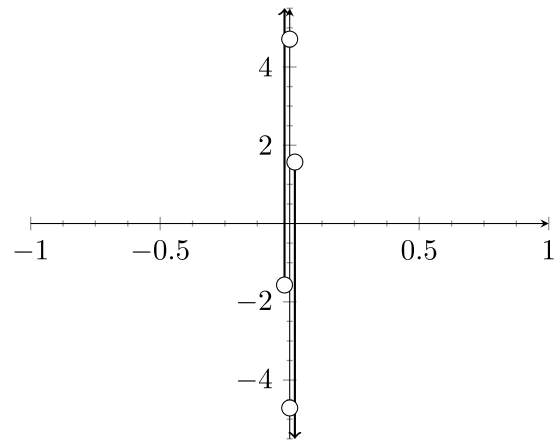

Figure 10.2: Stokes lines for the crystal growth problem. The Stokes lines and singularities have been horizontally shifted slightly for visibility.

once the function for ϕ_0 in (10.9) is used. Thus we obtain that

$$Q = \Lambda\sqrt{\cosh s}, \tag{10.19}$$

and the late terms are given by

$$\phi_n \sim \frac{\Lambda\sqrt{\cosh s}\,\Gamma(2n+\gamma)}{[\pm \mathrm{i}(s-\sigma_k)]^{2n+\gamma}} = \frac{\Lambda(-1)^n\sqrt{\cosh s}\,\Gamma(2n+\gamma)}{(s-\sigma_k)^{2n+\gamma}}. \tag{10.20}$$

It remains to determine γ and the constant Λ. These are done by matching near the singularities.

10.2 MATCHING NEAR THE SINGULARITY

Let us focus on the nearest singularity in the upper half-plane, $\sigma = \pi\mathrm{i}/2$; the analysis is analogous for other singularities. Firstly, we have the fact that

$$\phi_1 \sim c(s-\sigma)^{-2}, \tag{10.21}$$

$$\sqrt{\cosh s} \sim \mathrm{e}^{\pi\mathrm{i}/4}(s-\sigma)^{1/2}, \tag{10.22}$$

for some constant c. Then in the limit that $s \to \sigma$, the late terms follow

$$\begin{aligned} \epsilon^{2n}\phi_n &\sim \epsilon^{2n}\frac{\Lambda(-1)^n\mathrm{e}^{\pi\mathrm{i}/4}(s-\sigma)^{1/2}\Gamma(2n+\gamma)}{(s-\sigma)^{2n+\gamma}}, \\ &= \mathcal{O}\Big(\epsilon^{2n}(s-\sigma)^{1/2-2n-\gamma}\Big), \end{aligned} \tag{10.23}$$

which must be chosen to match the order of $\epsilon^2\phi_1$ at $n=1$. Hence we conclude that

$$\gamma = 1/2. \tag{10.24}$$

It remains to solve for Λ, and this must be done numerically. First, the size of the inner region can be guessed by examining the required size of $\mathcal{O}(s-\sigma)$ so that (10.11) is $\mathcal{O}(1)$. With $\gamma = 1/2$, we see that $s - \sigma = \mathcal{O}(\epsilon)$ is the inner region scaling. Thus let us set

$$s - \sigma = \epsilon z, \tag{10.25}$$

for the inner-region coordinate, $z = \mathcal{O}(1)$. Next, we examine the leading-order solution (10.9) in the limit $s \to \sigma$, noting that

$$\phi_0 \sim -\frac{\pi}{2} + \mathrm{i}\log(2/\epsilon) - \mathrm{i}\log(-z) + \mathcal{O}(\epsilon^2), \tag{10.26}$$

so we shall set

$$\phi(s) = -\frac{\pi}{2} + \mathrm{i}\log\left(\frac{2}{\epsilon}\right) - \mathrm{i}\psi(z). \tag{10.27}$$

Converting $\cos\phi$ to the equivalent expressions with complex exponentials, and then using (10.25) and (10.27) in (10.1a), we obtain after simplification

$$\psi''' + \psi' = -\mathrm{e}^{-\psi} + \frac{\epsilon^2}{4}\mathrm{e}^{\psi}, \tag{10.28}$$

and hence seeking the leading-order inner solution, $\psi \sim \psi_0$, we have

$$\psi_0''' + \psi_0' = -\mathrm{e}^{-\psi_0}. \tag{10.29}$$

Returning to (10.23) and using the inner-region scaling (10.25), we have

$$\epsilon^{2n}\phi_n \sim \frac{\Lambda \mathrm{e}^{\pi\mathrm{i}/4}\Gamma(2n+1/2)}{z^{2n}}, \tag{10.30}$$

which must be matched by Van-Dyke's rule to the outer limit of the inner expansion. This hints that we should expand:

$$\psi_0 \sim \log(-z) + \sum_{n=1}^{\infty}\frac{A_n}{z^{2n}}, \tag{10.31}$$

in the limit that $z \to \infty$.

However, this is almost as far as we are able to go analytically. When we substitute (10.31) into (10.29), we see that although the linear left-hand side can be written into a form that makes extracting the order-by-order values of z^{-2n} possible, the transcendental nature of the $\mathrm{e}^{-\psi}$ on the right-hand side makes it impossible to write down an explicit recursion relation for A_n.

Nevertheless, the first few orders can be derived by hand, and these yield, for A_1 to A_5:

$$(A_k) = \left\{2, -\frac{50}{3}, \frac{6104}{15}, -\frac{6197236}{315}, \frac{497432416}{315}, \ldots\right\}. \tag{10.32}$$

The alternating-sign divergent nature of the coefficients is clear.

Matching (10.23) to (10.31) and using (10.27), we see that,

$$-\mathrm{i}A_n \sim \Lambda(-1)^n \mathrm{e}^{\pi\mathrm{i}/4}\Gamma(2n+1/2), \tag{10.33}$$

and hence we may calculate Λ by numerically calculating

$$\Lambda = \mathrm{e}^{-\pi\mathrm{i}/4} \underbrace{\lim_{n\to\infty}\frac{A_n(-1)^{n+1}}{\Gamma(2n+1/2)}}_{b_n}. \tag{10.34}$$

A numerical script to compute the approximation to Λ using terms of A_n is given in below. We find that

$$\Lambda = \mathrm{e}^{-\pi\mathrm{i}/4}\Omega, \tag{10.35}$$

where $\Omega \approx 1.354$.

Numerical approximation of Λ

With $N = 50$, the last value returned is $b_{50} \approx 1.354$. However, the convergence can be accelerated using Richardson's extrapolation. For example, with the implementation shown in the script, the extrapolated value of

$$b_n \approx 1.354 \qquad (10.36)$$

is obtained, correct to the given digits.

```
Clear[A, b];
M = 50;
psi[z_] = Log[-z] + Sum[A[n]/z^(2 n), {n, 1, M}];
eqn = psi'''[z] + psi'[z] + Exp[-psi[z]];
Do[
  A[k] = A[k] /.
    First[Solve[SeriesCoefficient[eqn, {z, Infinity, 2*k +
        1}] == 0,
      A[k]]],
  {k, 1, M}];
Table[A[k], {k, 1, 5}]
b = Table[A[k]*(-1)^(k + 1)/Gamma[2 k + 1/2], {k, 1, M}];
Q0[n_, NN_] :=
  Sum[b[[n + k]]*(n + k)^NN*(-1)^(k + NN)/Factorial[k]/
     Factorial[NN - k], {k, 0, NN}];
qb1 = Table[N[Q0[k, 1]], {k, 1, M - 1}];
qb2 = Table[N[Q0[k, 2]], {k, 1, M - 2}];
qb3 = Table[N[Q0[k, 3]], {k, 1, M - 3}];
ListPlot[{b, qb1, qb2}]
Last[b] // N
Last[qb1] // N
Last[qb2] // N
Last[qb3] // N
```

Listing 10.1: Mathematica code to numerically solve for the value of Λ.

The behaviour of the coefficients b_n is shown in fig. 10.3, along with the series with one application of Richardson's extrapolation.

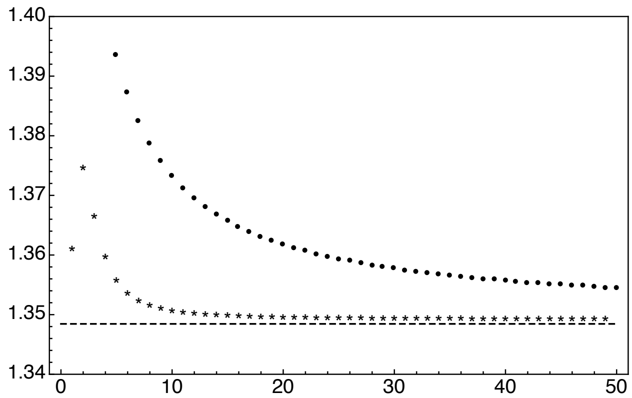

Figure 10.3: Behaviour of the coefficients b_n in (10.34) for the first fifty terms (filled circles). Also shown are the coefficients with one application of Richard's extrapolation applied (stars). The values converge to 1.349 . . . shown dashed.

Optimal truncation is found to be

$$\left|\frac{\epsilon^{2N+2}\phi_{N+1}}{\epsilon^{2N}\phi_N}\right| \sim \frac{4\epsilon^2 N^2}{|\chi|^2} \sim 1 \tag{10.37}$$

so we select

$$N = \frac{|\chi|}{2\epsilon} + \rho, \tag{10.38}$$

where ρ is bounded.

We optimally truncate

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n}\phi_n + R_N. \tag{10.39}$$

Substituting into (10.1a), we obtain to leading-order for the remainder equation:

$$\mathcal{L}R_N \equiv \epsilon^2 R_N''' + R_N' + R_N \sin\phi_0 \sim -\epsilon^{2N}\phi_{N-1}'''. \tag{10.40}$$

Note that $\phi_{N-1}''' \sim -\phi_N'$, so that this can be written as

$$\mathcal{L}R_N \sim \frac{\epsilon^{2N}(-\chi')Q(s)\Gamma(2N+\gamma+1)}{[\chi(s)]^{2N+\gamma+1}}. \tag{10.41}$$

Crucially, we note that $Q\mathrm{e}^{-\chi/\epsilon}$ is a solution to the homogenous equation $\mathcal{L}R_N = 0$. Consequently, we set the ansatz

$$R_N(s) = \mathcal{S}(s)Q(s)\mathrm{e}^{-\chi(s)/\epsilon}, \tag{10.42}$$

and seek to develop an equation for $\mathcal{S}$ as the Stokes line is crossed. When substituting (10.42) into (10.41), the terms that are proportional to $\mathcal{S}$ on the left hand-side sum to zero by design; at leading order, only terms that produce the highest powers of $1/\epsilon$ remain, yielding

$$\mathcal{L}R_N \sim -2\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}s}Q\mathrm{e}^{-\chi/\epsilon}. \tag{10.43}$$

We change to a coordinate frame local to the Stokes line and consider the rate-of-change of $\mathcal{S}$ as the Stokes line, $\operatorname{Im}\chi = 0$, $\operatorname{Re}\chi \geq 0$ is crossed. Letting $\chi = r\mathrm{e}^{\mathrm{i}\theta}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}\chi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\chi} = \frac{\chi'}{\mathrm{i}\chi}\frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{10.44}$$

Combining (10.41) with (10.43), we have

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\theta} \sim \epsilon^{2N}\left(\frac{\mathrm{i}}{2}\right)\frac{\Gamma(2N+\gamma+1)\mathrm{e}^{\chi/\epsilon}}{\chi^{2N+\gamma}}. \tag{10.45}$$

Now a simplification to the Gamma function using Stirling's approximation similar to that of (6.23) is applied. Using the optimal truncation point (10.38), we find that

$$\Gamma(2N+\gamma+1) \sim \sqrt{2\pi}\left(\frac{r}{\epsilon}\right)^{2N+\gamma+1/2}\left[\mathrm{e}^{-r/\epsilon}(1+\mathcal{O}(\epsilon)\right]. \tag{10.46}$$

Substitution into (10.45) gives

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\theta} \sim \left(\frac{\mathrm{i}}{2}\right)\sqrt{2\pi}\frac{r^{1/2}}{\epsilon^{\gamma+1/2}}\frac{\mathrm{e}^{-r/\epsilon}\mathrm{e}^{\chi/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(2N+\gamma)}}. \tag{10.47}$$

The exponentials are now combined. The key factor is in the numerator, with $\mathrm{e}^{-(r+\chi)/\epsilon}$, which is typically exponentially small unless $\chi = r\mathrm{e}^{\mathrm{i}\theta}$ is maximised, near $\theta = 0$. Note that

$$-\frac{r}{\epsilon}+\frac{r}{\epsilon}\left[1+\mathrm{i}\theta-\frac{\theta^2}{2}+\mathcal{O}(\theta)^3\right]-\mathrm{i}\theta\left[\frac{r}{\epsilon}+\gamma+2\rho\right] \sim -\frac{r\theta^2}{2\epsilon}+\mathcal{O}(\theta), \tag{10.48}$$

so the dominant balance occurs if $\theta = \mathcal{O}(\sqrt{\epsilon})$. We thus re-scale $\theta = \sqrt{\epsilon}\vartheta$, and obtain

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\vartheta} \sim \left(\frac{\mathrm{i}}{2}\right)\frac{\sqrt{2\pi r}}{\epsilon^{\gamma}}\mathrm{e}^{-r\vartheta^2/2}. \tag{10.49}$$

Integrating the above quantity from $\vartheta = -\infty$ to $\vartheta = \infty$ then yields the result that the jump in $\mathcal{S}$ about the Stokes line is

$$\left[\mathcal{S}\right] \sim \left(\frac{\mathrm{i}}{2}\right)\frac{\sqrt{2\pi r}}{\epsilon^{\gamma}}\int_{-\infty}^{\infty}\mathrm{e}^{-r\vartheta^2/2}\,\mathrm{d}\vartheta = \left(\frac{\mathrm{i}}{2}\right)\frac{\sqrt{2\pi r}}{\epsilon^{\gamma}}\sqrt{\frac{2\pi}{r}} = \frac{\pi\mathrm{i}}{\epsilon^{\gamma}}. \tag{10.50}$$

Altogether, we have shown that the jump in the optimally truncated solution across the Stokes line from the singularity at $\sigma = \pi\mathrm{i}/2$ is given by

$$\phi_{\text{exp}} \sim \frac{\pi\mathrm{i}}{\epsilon^{\gamma}}\Lambda\sqrt{\cosh s}\mathrm{e}^{-\chi/\epsilon}. \tag{10.51}$$

Using the fact that $\gamma = 1/2$, $\Lambda = \Omega\mathrm{e}^{-\pi\mathrm{i}/4}$ from (10.35), and $\chi = \mathrm{i}(s-\pi\mathrm{i}/2)$, we have the final prediction of

$$\phi_{\text{exp}} \sim \frac{\pi}{\epsilon^{1/2}}(1.35\ldots)\mathrm{e}^{\pi\mathrm{i}/4}\sqrt{\cosh s}\mathrm{e}^{-\chi/\epsilon}. \tag{10.52}$$

Author's note I

The above is a very finicky calculation that we have not yet managed to match the original, as presented in Chapman et al. [1998b]. Their result is that the contribution from the $s = \pi\mathrm{i}/2$ singularity is

$$\sim -\frac{\tilde{\Lambda}\pi\mathrm{e}^{-\pi\mathrm{i}/4}\sqrt{\cosh s}\mathrm{e}^{-\pi/(2\epsilon)}\mathrm{e}^{-\mathrm{i}s/\epsilon}}{\epsilon^{1/2}}, \tag{10.53}$$

where they later find $\tilde{\Lambda} = \Omega\mathrm{e}^{\pi\mathrm{i}/4}$ with $\Omega \approx 1.35$. So we are a factor of $-\mathrm{e}^{\pi\mathrm{i}/4}$ off. It is plausible that this issue arises during the inner matching procedure.

Author's note II

One route to bypass the painful optimal truncation and Stokes-line smoothing procedure is to apply the formal Borel re-summation method presented previously. This appears in the exercise leading to (9.16).

10.4 VARIATIONS OF THE CRYSTAL GROWTH PROBLEM

There are variations of the classic geometric crystal growth problem that are worth discussing.

The first is the case of anisotropic crystal growth problem, as presented in Kruskal and Segur [1991], Chapman et al. [1998b], which modifies the differential equation to

$$\epsilon^2 \phi''' + \phi' = \frac{\cos\phi}{1 + \alpha\cos(4\phi)}. \tag{10.54}$$

In this case, the introduction of the anisotropic parameter, $0 < \alpha < 1$ mainly changes the singularity structure in the complex plane. Now, there are singularities in each of the four quadrants. These singularities coalesce together, in pairs, at the previous locations, $s = \pm\pi\mathrm{i}/2$ in the limit $\alpha \to 0$, while in the limit $\alpha \to 1$ they descend to the real axis. Stokes lines, however, are still linear; the difference with the new arrangement of singularities is that there now exists a countably infinite number of ϵ values where it is possible the exponentially-small switchings are confined to a local region around $s = 0$.

10.5 EXERCISES

1. By setting $f = A\mathrm{e}^{ms}$ into (10.2), develop the characteristic polynomial for $m = m(\epsilon)$ and determine the leading-order behaviour of the respective roots for small values of ϵ. Conclude with the form (10.3).
2. Study the application of Richardson's extrapolation to accelerate the series convergence of the b_n series. In particular, with the assumption that

$$b_n \sim Q_0 + \frac{Q_1}{n} + \mathcal{O}(1/n^2), \tag{10.55}$$

we use the formula

$$Q_0 \sum_{k=0}^{N} \frac{b_{n+k}(n+k)^N(-1)^{k+N}}{k!(N-k)} + \mathcal{O}\left(\frac{1}{n^{N+1}}\right). \tag{10.56}$$