9 ON THE UNIVERSALITY OF EXPONENTIAL ASYMPTOTICS

This chapter will be left somewhat unfinished until later; I need to decide what are the crucial details to mention! For the most part, the reader can proceed to the later chapters, and this chapter can be re-edited and back-referenced as we continue deeper in the material.

`Draft chapter last generated 2024-12-02; P.H. Trinh`

The practitioner understands that, at least concerning the practical application of asymptotic analysis, a great many problems in applications are studied using the same 'recipe': given the typical perturbative problem involving a small parameter ϵ, one attempts to develop solutions in an expansion in powers of ϵ. All problems involve such a recipe—but all problems are also different!

In relation to exponential asymptotics, there are similar recipes and universalities in the typical problems. In this chapter, we attempt to provide a succinct guide to the general procedure. Significant liberties will be taken, but whenever possible, we shall mention example problems where the rules may need to be modified.

9.1 FORMAL BOREL SUMMATION

The following is a brief note on the formal manipulation of the tail of the divergent series, demonstrating the association of factorial-over-power divergence with the emergence of exponentials. This serves well to bypass the more heavy-handed route of Stokes-line smoothing introduced previously.

Consider the expression of the remainder, where we assume that the late terms are entirely approximated by a factorial-over-power form:

$$R_N(z) \sim \sum_{n=N}^{\infty} \frac{\epsilon^n Q(z)\Gamma(n+\gamma)}{\chi^{n+\gamma}}. \tag{9.1}$$

We replace the gamma function by its integral definition:

$$\begin{aligned} R_N(z) &\sim \sum_{n=N}^{\infty} \frac{\epsilon^n Q(z)}{\chi^{n+\gamma}} \int_0^{\infty} \mathrm{e}^{-t} t^{n+\gamma-1}\,\mathrm{d}t \\ &= \frac{Q(z)}{\epsilon^{\gamma}} \sum_{n=N}^{\infty} \int_0^{\infty} \mathrm{d}t \cdot \frac{\mathrm{e}^{-t}}{t} \left(\frac{\epsilon t}{\chi}\right)^{n+\gamma}. \end{aligned} \tag{9.2}$$

The assumption is made that that operations of summation and integration can be interchanged:

$$R_N(z) \sim \frac{Q(z)}{\epsilon^{\gamma}} \int_0^{\infty} \mathrm{d}t \frac{\mathrm{e}^{-t}}{t} \sum_{n=N}^{\infty} \left(\frac{\epsilon t}{\chi}\right)^{n+\gamma}, \tag{9.3}$$

and we shall not think too carefully about the convergence of the geometric sum:

$$R_N(z)\frac{Q(z)}{\epsilon^\gamma}\int_0^\infty \mathrm{d}t\frac{\mathrm{e}^{-t}}{t}\frac{(\epsilon t/\chi)^{N+\gamma}}{1-\epsilon t/\chi}. \tag{9.4}$$

It is convenient now to set $\epsilon t = w$ and hence the integral can be written as

$$R_n(z) \sim \frac{Q(z)}{\epsilon^\gamma}\int_0^\infty \frac{w^{N+\gamma-1}}{\chi^{N+\gamma}}\frac{\mathrm{e}^{-w/\epsilon}}{1-w/\chi}\,\mathrm{d}w. \tag{9.5}$$

Now we consider an initial choice of z such that the singularity in the w-plane, which happens at $w = \chi(z)$, lies in the lower half-plane. The integration along the domain of $w \in [0, \infty)$ proceeds as usual, and integration by parts will essentially recover the algebraic corrections from the asymptotic expansion. However, if z is analytically continued so that the pole at $t = \chi(z)$ crosses the positive real w-axis, then we see that a residue contribution must be included which is equivalent to the local integral:

$$\sim \frac{Q(z)}{\epsilon^\gamma}\frac{1}{\chi^{N+\gamma}}\oint(-\chi)w^{N+\gamma-1}\frac{\mathrm{e}^{-w/\epsilon}}{w-\chi}\,\mathrm{d}w = \frac{2\pi\mathrm{i}Q(z)}{\epsilon^\gamma}\mathrm{e}^{-\chi/\epsilon}. \tag{9.6}$$

This is a demonstration through formal re-summation that if the late terms of an asymptotic expansion are given by $Q(z)\Gamma(n+\gamma)/\chi^{n+\gamma}$, then they are associated with the Stokes line switching given above.

9.2 THE FACTORIAL-OVER-POWER METHODOLOGY

Consider a generic nonlinear differential equation on $y = y(z)$:

$$\mathcal{N}(z, y, y', y'', \ldots; \epsilon) = 0. \tag{9.7}$$

Observations of the universality:

1. Singular asymptotic expansions diverge according to a factorial-over-power.
2. There exists Stokes lines. These are given where one expansion reaches peak exponential dominance over another.
3. Across Stokes lines, there is the Stokes Phenomenon.

Definition 9.1 (The Darboux property)
Consider a singular differential equation for $y(z)$ in the limit $\epsilon \to 0$. If the regular perturbative expansion can be written as[1]

$$y(z) \sim \sum_{n=0}^\infty \epsilon^n y_n, \tag{9.8}$$

then in the limit $n \to \infty$, the divergence takes the form of a factorial-over-power:

$$y_n \sim \frac{A\Gamma(n+\gamma)}{\chi^{n+\gamma}}, \tag{9.9}$$

where χ, A, and γ are in general functions of $z \in \mathbb{C}$.

[1] We have given a specific example where the perturbative expansion proceeds in powers of ϵ. In cases where it proceeds in powers of, for instance ϵ^2 this will modify the factoril-over-power form of the divergence. See examples in later chapters.

Refs: Dingle [1973], Chapman et al. [1998b]

Definition 9.2 (Stokes lines)
Consider two asymptotic approximations for (9.7)

$$\begin{aligned} y_1 &= (A_0(z) + \epsilon A_1(z) + \ldots)\, \mathrm{e}^{-\chi_1(z)/\epsilon}, \\ y_2 &= (B_0(z) + \epsilon B_1(z) + \ldots)\, \mathrm{e}^{-\chi_2(z)/\epsilon}. \end{aligned} \tag{9.10a}$$

A Stokes line consists of a curve, $\gamma_s \in \mathbb{C}$, such that, when one asymptotic dominant expansion, say y_1 is analytically continued across γ_s, the sub-dominant expansion switches on (according to the Stokes Phenomenon). In this case, we write

$$y_1 \xrightarrow{y_1 > y_2} y_1 + \mathcal{S}(z) y_2. \tag{9.11}$$

This occurs when

$$\mathrm{Im}[\chi_1](z) = \mathrm{Im}[\chi_2](z) \qquad \mathrm{Re}[\chi_1](z) \geq \mathrm{Re}[\chi_2](z). \tag{9.12}$$

Definition 9.3 (Optimal truncation)
The rule-of-thumb for optimal truncation is that it occurs where adjacent terms of the asymptotic sequence is approximately equal. Thus for the series (9.8), this is the point $n = N$ where

$$\left| \frac{\epsilon^N y_N}{\epsilon^{N-1} y_{N-1}} \right| \sim 1. \tag{9.13}$$

In specific cases it can be justified *a priori* Berry and Howls 1990, Berry 1991b, Costin and Kruskal 1999.

Definition 9.4 (Stokes smoothing)
Stokes smooth typically occurs in the form of an error function switching.

9.3 FURTHER INTERESTING AND SUBTLE EFFECTS

- Stokes lines have finite width and at large distances from the singularity, this finite-width property can cause Stokes lines to intersect and interact. It is possible for Stokes lines running parallel or asymptotic to the real axis to interact with the real axis at large distances [King, 1998, Chapman and King, 2003].
- Stokes lines generated by asymptotically close singularities can be viewed as a single Stokes line from far away, but require the resolution of complicated asymptotics near the singularities [Trinh and Chapman, 2015].
- Crossing Stokes lines are often associated with: (i) the higher-order Stokes Phenomenon; (ii) the appearance of Stokes lines from singularities that are not present in any low-order term; (iii) Stokes lines that stop at a point.
- The factorial-over-power ansatz is not universal [Trinh and Chapman, 2015].
- The error-function smoothing is not universal [Chapman, 1996].
- Stokes lines can 'pile-up'. The dominant contribution of Stokes lines near the origin is not from the nearest singularity but from a clustering singularities at infinity [Chapman et al., 2013].

9.4 EXERCISES

1. Given an asymptotic expansion of the form

$$y(z) \sim \sum_{n=0}^{\infty} \epsilon^{2n} y_n(z), \tag{9.14}$$

suppose that the late terms diverge in the form of

$$y_n(z) \sim \frac{Q(z)\Gamma(2n+\gamma)}{[\chi(z)]^{2n+\gamma}}, \tag{9.15}$$

where γ is constant. Demonstrate using the formal Borel summation procedure that the above divergent series is associated with an exponentially-small switching of the form

$$y_{\exp} \sim \frac{\pi \mathrm{i}}{\epsilon^{\gamma}} Q(z) \mathrm{e}^{-\chi(z)/\epsilon}. \tag{9.16}$$

Therefore, the main difference due to the expansion going up in powers of ϵ^2, and also the $\Gamma(2n+\gamma)$ trend is a halving of the prediction (9.6).