`Draft chapter last generated 2024-12-02; P.H. Trinh`

7

George Gabriel Stokes' struggle to understand what is today known as the Stokes phenomenon is famously documented in his correspondence with his fiancée, Mary Susanna Robinson.[1] Stokes mentions an "integral of Airy's"—this corresponds to an expression previously derived by Airy to describe the intensity of light near a caustic [cf. Airy [1838]].

[1]*"When the cat's away the mice may play. You are the cat and I am the poor little mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it*. I took it up again a few days ago, and after a two or three day's fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it. . . . . " –March 1857, [Stokes, 1907, p. 62]*

In order to illustrate the nature of the phenomenon Stokes references, we will find it more instructive to consider the differential equation formulation instead.

> *Airy equation*
>
> The classic real-valued Airy equation is defined on the real x-axis, with decaying conditions at infinity:
>
> $$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - xf = 0, \tag{7.1a}$$
>
> $$f \to 0 \quad \text{as } x \to \pm\infty. \tag{7.1b}$$

The second-order differential equation has been defined with decay conditions at infinity, and there the freedom to specify an additional normalisation condition, say the value of $f(0)$. This shall be specified later.

In order to approximate solutions, series expansions must be used; Stokes recognised that the typical convergent series expansions near $x = 0$ are too slowly convergent to be useful [Stokes, 1851], and it is better to consider asymptotic expansions as $|x| \to \infty$. In this limit, it is convenient to re-scale $x^{3/2} = z^{3/2}/\epsilon$ so that we have instead

$$\epsilon^2 \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - zf = 0, \tag{7.2}$$

with $\epsilon \to 0$.

With the intuition that solutions decay exponentially as $z \to \infty$, let us search for the solution in terms of a Liouville-Green or WKBJ ansatz, $f \sim A(z)\mathrm{e}^{-\chi(z)/\epsilon}$. Substitution into the above equation, we find by matching to the first two orders of ϵ that

$$\chi(z) = \pm\frac{2}{3}z^{3/2} \quad \text{and} \quad A(z) = \frac{\text{const.}}{z^{1/4}}. \tag{7.3}$$

Since the solution decays on the positive real axis by (7.1b), then the WKBJ approximation is taken to be the single mode,

$$f(z) \sim \frac{\mathcal{A}}{z^{1/4}}\mathrm{e}^{-\frac{2}{3}z^{3/2}/\epsilon}. \tag{7.4}$$

where $\mathcal{A} \in \mathbb{R}$. In writing the functions $z^{1/4}$ and $z^{3/2}$ above, we have consider the principal branches, with the branch cuts taken along $\operatorname{Arg} z = 2\pi$.

The WKBJ solution (7.4) fails to be valid at $z = 0$, known as the *turning point*. However, let us consider approximating f with (7.4) as z is analytically continued in the upper-half complex plane—from large and positive values of z to large and negative z. As we do so, the WKBJ solution remains well-defined, and there is no sign that it would fail to approximate the true solution. Yet on the negative real axis, (7.4) now evaluates to non-zero imaginary numbers, but we have assumed that the differential equation is entirely real. There must be more going on.

For $z < 0$, let us assume that the solution is composed of the two linearly independent components related to (7.3), and write

$$f(z) \sim \frac{\mathcal{A}}{z^{1/4}} e^{-\frac{2}{3} z^{3/2}/\epsilon} + \frac{\mathcal{B}}{z^{1/4}} e^{\frac{2}{3} z^{3/2}/\epsilon}. \tag{7.5}$$

Thus we have assumed that the first exponential has the same prefactor, $\mathcal{A}$, as it did along the positive real axis. On the negative real axis, $z^{3/2}$ is purely imaginary while $z^{1/4} = e^{\pi i/4}|z|^{1/4}$. The only way in which (7.6) can be purely real and bounded as $\epsilon \to 0$ with $z < 0$ is if $\mathcal{B} = i\mathcal{A}$. Simplifying yields

$$f(z) \sim \frac{2\mathcal{A}}{|z|^{1/4}} \cos\left(\frac{2}{3\epsilon}|z|^{3/2} + \frac{\pi}{4}\right), \tag{7.6}$$

which thus predicts an real-valued oscillatory solution with an algebraically decaying amplitude.

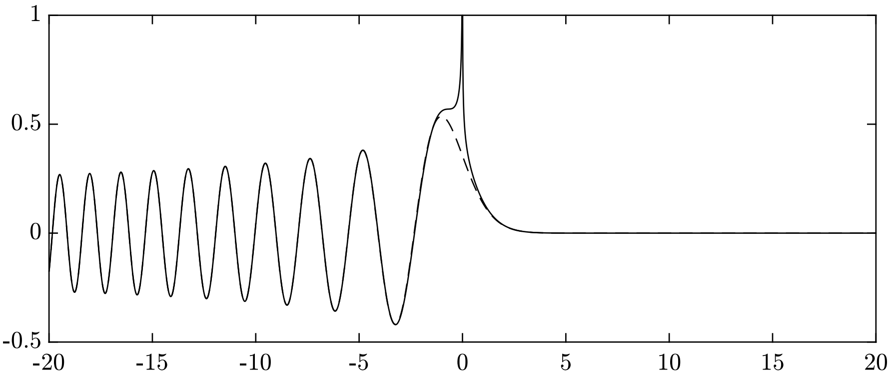

Figure 7.1: The exact $\operatorname{Ai}(x)$ (shown dashed) compared to its leading-order asymptotic approximation (7.7) developed as $x \to \pm\infty$ (shown solid).

Recalling that $z = x\epsilon^{2/3}$, the *Airy function of the first kind*, $\operatorname{Ai}(x)$, is defined so that the constant $\mathcal{A} = 1/(2\sqrt{\pi})$. With this choice of normalisation, the leading-order asymptotic approximation to the Airy equation (7.1a) is $f(x) = \operatorname{Ai}(x)$, with

$$\operatorname{Ai}(x) \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}x^{\frac{1}{4}}} e^{-\frac{2}{3}x^{\frac{3}{2}}} & \text{as} \quad x \to \infty, \\ \dfrac{1}{2\sqrt{\pi}x^{\frac{1}{4}}} \left(e^{-\frac{2}{3}x^{\frac{3}{2}}} + i e^{\frac{2}{3}x^{\frac{3}{2}}}\right) & \text{as} \quad x \to -\infty. \end{cases} \tag{7.7}$$

This leading-order approximation is shown in fig. 7.1, and we see that the fit of the leading approximation to the exact value is excellent, even for only moderately large values of x.

By now, we recognise many of the telltale signs of exponential asymptotics. Stokes had understood that the presence of the previously subdominant exponential, with $\mathcal{B}$ in (7.6), was necessary based on the boundary conditions on the negative real axis. But at the time, there is no reason *a priori* to have believed that the secondary exponential should be included once the leading-order solution (7.4) is analytically continued from the right to the left. Indeed, if the secondary exponential is included on the left, then it must have been *switched-on* at some point during the path of continuation—and yet there is no clear sign of this. This is the conundrum that occupied Stokes for a number of years, as referenced in the earlier letter.

7.1 INTEGRAL FORMULATION OF THE AIRY EQUATION

As with our examination in the previous chapter, it is advantageous to consider the integral formulation of the Airy problem; indeed the integral formulation was the original focus of Stokes' work. We define the Fourier and inverse Fourier transforms, respectively, as [Howison, 2005, p. 146]:

> *Fourier and Inverse Fourier transforms*
>
> For a suitably smooth and integrable function f, we define the Fourier transform, $\hat{f}$, and inverse Fourier transform of f via
>
> $$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x, \tag{7.8a}$$
>
> $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{7.8b}$$

Using the standard results, applying the Fourier transform to (7.1a) gives

$$\mathrm{i}\frac{\mathrm{d}\hat{f}}{\mathrm{d}k} - k^2\hat{f} = 0. \tag{7.9}$$

We have that $\hat{f'(x)} = -\mathrm{i}k\hat{f}$ established by integration by parts, and $\hat{xf} = -\mathrm{i}\hat{f}'(k)$, established by differentiation under the integral.

Solving for $\hat{f}$ and inverting gives

$$f(x) = \frac{A}{2\pi\mathrm{i}}\int_C \exp\left(-\frac{s^3}{3} + sx\right)\,\mathrm{d}s, \tag{7.10}$$

for a constant of integration A, where we have transformed the integration variable from k to $s = -\mathrm{i}k$. The infinite contour, C in (7.10) must proceed from one sector of the complex s-plane to another, in such a way that the integral converges as $|s| \to \infty$. Hence we require $\mathrm{Re}\{s^3\} \to \infty$ as $|s| \to \infty$. Letting $\varphi = \operatorname{Arg} s$, we thus require that s goes to infinity within one of the three possible sectors

$$-\pi/6 < \varphi < \pi/6, \qquad \pi/2 < \varphi < 5\pi/6, \qquad -5\pi/6 < \varphi < -\pi/2. \tag{7.11}$$

Hence C is chosen to start at infinity in one of the above sectors, and end at infinity in one of the other two. Of the six possibilities, two linearly independent solutions can be defined by the contours shown in fig. 7.2.

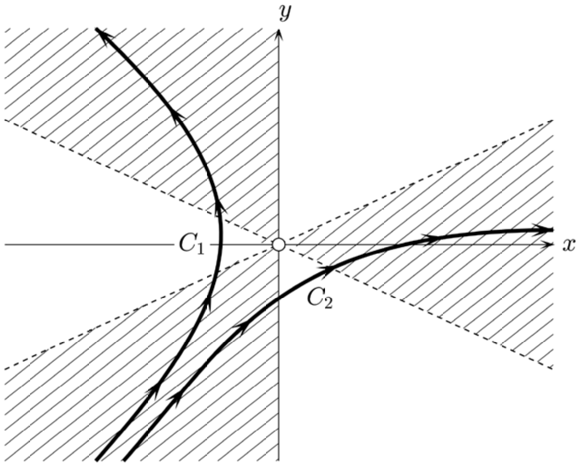

Figure 7.2: Two possible contours for Airy equation solutions. The contour C_1 is used to define the Ai function and the contour C_2 used to define the Bi function. The hatched wedges have interior angles of $2\pi/6$.

When A in (7.10) is taken as $A = 1$, the contour that runs from $\infty e^{-2\pi i/3}$ to $\infty e^{2\pi i/3}$ defines the Airy function of the first kind, denoted Ai[2]. Similarly, the contour running from $\infty e^{-2\pi i/3}$ to ∞ defines the Airy function of the second kind, denoted Bi. In summary,

[2] Indeed the normalisation constant mentioned following (7.1) is chosen so that the solution agrees with the scaling of the Ai function.

$$\mathrm{Ai}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\infty e^{-\frac{2\pi i}{3}}}^{\infty e^{\frac{2\pi i}{3}}} \exp\left(-\frac{s^3}{3} + sx\right) \mathrm{d}s, \tag{7.12a}$$

$$\mathrm{Bi}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\infty e^{-\frac{2\pi i}{3}}}^{\infty} \exp\left(-\frac{s^3}{3} + sx\right) \mathrm{d}s. \tag{7.12b}$$

Choices of contours from the other sectors may be transformed via a rotation (*e.g.* $x \mapsto e^{2\pi i/3}x$) onto one of those used above, so we see that the above two are the only linearly independent solutions. We shall focus on $\mathrm{Ai}(x)$.

7.2 THE METHOD OF STEEPEST DESCENTS

We wish to apply the method of steepest descents in order to derive the asymptotic approximation of the integral (7.12a) in the limit of $|x| \to \infty$. Writing $x = re^{i\theta}$ and setting $\lambda = r^{3/2}$, we have now the integral

$$\mathrm{Ai}(x) = \frac{\lambda^{1/3}}{2\pi \mathrm{i}} \int_{\infty e^{-\frac{2\pi i}{3}}}^{\infty e^{\frac{2\pi i}{3}}} e^{\lambda w(t)}\, \mathrm{d}t, \tag{7.13}$$

obtained from changing $s = r^{1/2}t$, and where we have defined the exponential argument

$$w(t) = te^{i\theta} - \frac{t^3}{3}. \tag{7.14}$$

The only critical points of the integrand in (7.13) correspond to saddle points, where $w'(s) = 0$, and thus

$$t = t_{\pm} \equiv \pm e^{i\theta/2}. \tag{7.15}$$

Note that these are simple saddles, as $w''(t_{\pm}) \neq 0$. Therefore, as the argument of x changes in the $\mathrm{Ai}(x)$, the two saddle points rotate about the unit circle in the t-plane. The task is to determine how the deformation of the contour C proceeds dependent on the configuration of the saddle points.

The paths of steepest descent/ascent are specified by those values of $t \in \mathbb{C}$ where $\operatorname{Im} w(t)$ is constant, hence those through the two saddles are given by $\operatorname{Im} w(t) = \operatorname{Im} w(t_\pm)$. Letting $t = u + \mathrm{i}v$, this gives

$$-\left(u^2 v - \frac{v^3}{3}\right) + u\sin(\theta) + v\cos(\theta) = \pm\frac{2}{3}\sin\left(\frac{3\theta}{2}\right). \tag{7.16}$$

Thus, for a given $\operatorname{Arg} x = \theta$, contours of steepest descent through $t_\pm$ are found by solving the implicit equation (7.16) along with the requirement that the path descends in the appropriate manner (with descent characterised by $\operatorname{Re} w(t) \le \operatorname{Re} w(t_\pm) = \pm\frac{2}{3}\cos(3\theta/2)$).

For example, if $\theta = 0$, the two saddle points lie along $t_\pm = \pm 1$. The original contour of integration, shown as C_1 in fig. 7.2 will be deformed to pass through $t_- = -1$. This is shown in fig. 7.3. Our interest will be in examining the change in the paths of steepest descent as θ changes.

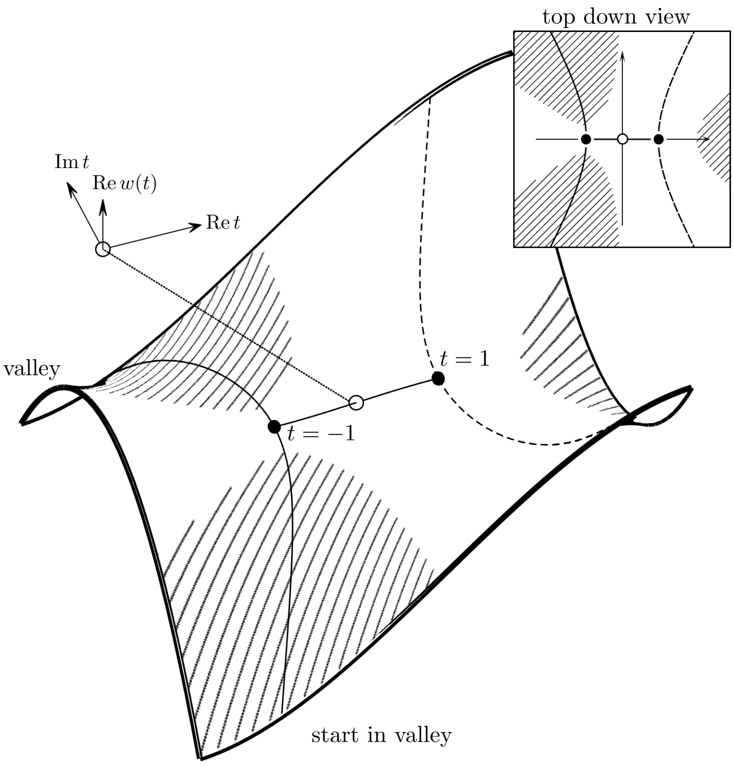

Figure 7.3: When $\theta = 0$, the two saddle points are located at $t = \pm 1$, shown in the contour plot of the t-plane of the inset. The path of integration should then be deformed to pass through $t = -1$. The three-dimensional surface plot shows $(\operatorname{Re} t, \operatorname{Im} t, \operatorname{Re} w(t))$. Valleys are shown hatched.

These paths are shown in fig. 7.4. The procedure is thus described as follows:

Figure 7.4a We begin along the positive real x-axis, with $\theta \equiv \operatorname{Arg}(x) = 0$. Then C should be chosen to pass through the saddle point at $t = t_-$; the standard computation of section 7.2.1 shows that the contribution from this point is

$$\operatorname{Ai}(x) \sim \mathcal{A} e^{-\frac{2}{3}x^{3/2}}, \quad \text{with} \quad \mathcal{A} = \left[\frac{1}{2x^{1/4}\sqrt{\pi}}\right], \tag{7.17}$$

once we have re-written in terms of x.

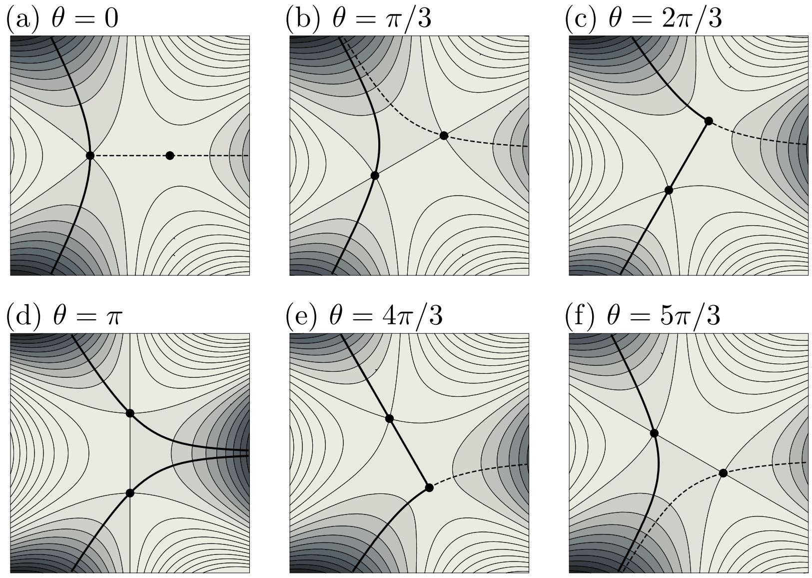

Figure 7.4: These figures illustrate the steepest descent contours chosen to approximate the Airy integral. The contour should begin at $\infty e^{-2\pi \mathrm{i}/3}$ and end at $\infty e^{2\pi \mathrm{i}/3}$. The saddle points are marked as nodes, and the thin lines are constant contours of $\mathrm{Re}[w(t)]$, with dark regions corresponding to valleys. The thick solid line(s) indicate the correct contours to follow. Dashed line(s) also indicate steep paths from the saddles that are ultimately unused. The top three subfigures are for $\theta = 0, \pi/3, 2\pi/3$ and the bottom three for $\theta = \pi, 4\pi/3, 5\pi/3$.

Figure 7.4b As we analytically continue $x = re^{\mathrm{i}\theta}$, note that for $\theta \in [0, 2\pi/3)$, the topology of the integration contour remains the same, and so (7.17) continues to hold throughout.

Figure 7.4c However at $\theta = 2\pi/3$, the descending contour from $t = t_-$ passes through the other saddle point at $t = t_+$. Thus, immediately thereafter, the approximation must include an exponentially small contribution. We write this transition through $\theta = 2\pi/3$ as

$$\mathcal{A}e^{-\frac{2}{3}x^{3/2}} \xrightarrow{\theta=2\pi/3} \mathcal{A}e^{-\frac{2}{3}x^{3/2}} + \mathrm{i}\mathcal{A}e^{\frac{2}{3}x^{3/2}}. \tag{7.18}$$

The fact that the Stokes phenomenon switches on the exponential with factor $\mathrm{i}\mathcal{A}$ is certainly not obvious, but can be argued either by examining the precise switching by steepest descents, or alternatively arguing on the basis that the solution profile must be oscillatory on the negative real axis.

Figure 7.4d–f Notice that now at $\theta = \pi$, the previously subdominant contribution from $t = t_+$ becomes the same size as the contribution from the original saddle, $t = t_-$, as evident from the equal values of $\mathrm{Re}\, w(t)$; this secondary contribution from $t = t_+$ continues increasing in size until reaching peak dominance at $\theta = 4\pi/3$, where the term in (7.17) now switches off.

As introduced in chapter 6, these switchings are occurrences of the Stokes phenomenon. Since the paths of steepest descent are curves of constant phase of $w(t)$, then if $\mathrm{Im}\, w(t_+) = \mathrm{Im}\, w(t_-)$, there exists a steepest descent or ascent path from one saddle to the other—*i.e.* the other saddle now has the possibility of 'switching on'. This is equivalent to the condition

$$\mathrm{Im}(\mathrm{e}^{3\mathrm{i}\theta/2}) = 0 \implies \theta = 2n\pi/3, \qquad n \in \mathbb{Z}. \tag{7.19}$$

For $0 \le \theta < 2\pi$ this occurs when $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Note that in order for one saddle, say t_- to switch on the second, then the path must descend from the initial saddle to the second saddle. For the first transition at $\theta = 2\pi/3$, this corresponds to the descent condition of $\mathrm{Re}\, w(t_-) \ge \mathrm{Re}\, w(t_+)$. Hence there is a Stokes line at $\theta = 2\pi/3$ where saddle t_- switches-on t_+. And there is a Stokes line at $\theta = 4\pi/3$ where t_+ switches-off saddle t_-.

We can equally define an *Anti-Stokes line* as the critical line where both critical points contribute equally to the asymptotic expansion. When such lines are crossed, the dominance of the two contributions is exchanged. This must happen where $\mathrm{Re}\, w(t_+) = \mathrm{Re}\, w(t_-)$, so where

$$\mathrm{Re}(\mathrm{e}^{3\mathrm{i}\theta/2}) = 0 \implies \theta = (2n-1)\pi/3, \qquad n \in \mathbb{Z}, \tag{7.20}$$

and for $0 \le \theta < 2\pi$ this occurs when $\theta = \pi/3$, π, $5\pi/3$.

Following an illustration proposed by Stokes, this procedure can be imagined through the following representation. In fig. 7.5, we plot three curves in the complex x-plane showing a representation of the exponential scalings of the two relevant contributions from t_- and t_+. The dashed curve is the unit circle. The solid curve, representing t_-, is $1 - \Delta\cos(3\theta/2)\mathrm{e}^{\mathrm{i}\theta}$ for θ ranging from 0 to 2π. The dash-dotted curve, representing t_+, is $1 + \Delta\cos(3\theta/2)\mathrm{e}^{\mathrm{i}\theta}$. For the illustration, $\Delta = 0.3$.

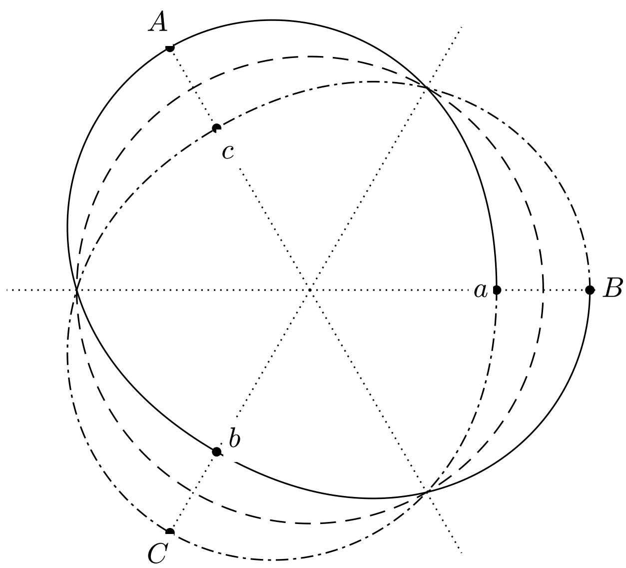

Figure 7.5: An illustration of the exponential dominance and sub-dominance shown in the complex plane. The solid line corresponds to the initially exponentially decaying solution from t_- while the dash-dotted line corresponds to the initially exponentially growing solution from $t = t_+$. The dashed line is a unit circle.

Thus our study of $\mathrm{Ai}(x)$ begins with $\theta = 0$ where the exponentially decaying solution is marked with 'a' on the solid line. The exponentially growing solution, marked 'B' is not present. As $\mathrm{Ai}(x)$ is analytically continued to $\mathrm{Arg}\, x = \theta = 2\pi/3$, the exponential present reaches peak exponential dominance at the point marked 'A', at which the subdominant solution switches on at the point marked 'c'. The switch occurs again at 'C' and 'b'.

7.2.1 Saddle contributions

We may verify the saddle-point contributions (7.17) and (7.18) derived in the previous section. Near $t = t_{\mp}$, we have

$$w(t) \sim \mp\frac{2}{3}\mathrm{e}^{\frac{3\mathrm{i}\theta}{2}} \pm \mathrm{e}^{\mathrm{i}\theta/2}(t \pm \mathrm{e}^{\mathrm{i}\theta/2})^2 + \cdots . \tag{7.21}$$

Consider the saddle at t_-. Substituting the above expansion into the integral expression (7.12a) and recalling that $z = r\mathrm{e}^{\mathrm{i}\theta}$, we have

$$I_- \sim \frac{\lambda^{\frac{1}{3}}\mathrm{e}^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\pi\mathrm{i}} \int_{\text{near } s_-} \exp\Big[\lambda \mathrm{e}^{\mathrm{i}\theta/2}(s + \mathrm{e}^{\mathrm{i}\theta/2})^2 + \cdots\Big]\, \mathrm{d}s. \tag{7.22}$$

Note we take the principal branch of the square root, and consider the branch cut along the positive real axis, with $0 \le \operatorname{Arg} z < 2\pi$. The integral will be evaluated along a small distance, say $-\delta$ to δ, along the steepest descent path about the saddle. Ignoring the higher-order terms not shown, we make the substitution $(s + \mathrm{e}^{\mathrm{i}\theta/2}) = R\mathrm{e}^{\mathrm{i}\nu}$ and approximate the path segment by a straight line with constant ν to obtain

$$I_- \sim \frac{\lambda^{\frac{1}{3}}\mathrm{e}^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\pi\mathrm{i}} \int_{-\delta}^{\delta} \exp\Big[\lambda \mathrm{e}^{\mathrm{i}\theta/2}R^2\mathrm{e}^{2\mathrm{i}\nu}\Big]\, \mathrm{e}^{\mathrm{i}\nu}\, \mathrm{d}R, \tag{7.23}$$

where $\lambda \to \infty$.

To find the necessary value of ν, we use the steepest descent criterion beginning at the saddle, where $R = 0$. This gives

$$\operatorname{Im}[R^2\mathrm{e}^{\mathrm{i}(2\nu+\theta/2)}] = 0 \implies \nu = \frac{n\pi - \theta/2}{2} \quad \text{where } n \in \mathbb{Z},$$
$$\operatorname{Re}[R^2\mathrm{e}^{\mathrm{i}(2\nu+\theta/2)}] \le 0 \implies n = \pm 1.$$

Thus, we set $\nu = \pi/2 - \theta/4$ for the steepest descent path.

The last step is to extend the integration from $-\delta$ to δ in R to $-\infty$ and ∞. This introduces only exponentially small errors. Hence to leading order in the approximation,

$$I_- \sim \frac{\lambda^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\theta/4}\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\pi} \int_{R=-\infty}^{\infty} \mathrm{e}^{-\lambda R^2}\, \mathrm{d}R \tag{7.24}$$

and thus via the Gaussian integration,

$$I_- \sim \mathcal{A}(x)\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}, \qquad \text{where } \mathcal{A}(x) = \frac{1}{2\sqrt{\pi}x^{\frac{1}{4}}}, \tag{7.25}$$

matching (7.17) for the standard decaying exponential initially valid for $x > 0$.

The analogous steepest descent procedure for the t_+ saddle gives

$$I_+ \sim -\frac{\mathrm{e}^{\frac{2}{3}x^{\frac{3}{2}} - \frac{\mathrm{i}\pi}{2}}}{2\sqrt{\pi}x^{\frac{1}{4}}} = \mathrm{i}\mathcal{A}(x)\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}, \tag{7.26}$$

which returns the second term of (7.18).

Remarks on steepest descent procedures

It is worth revising the steepest descent computation for the Airy problem as compared to the exponential integral and complementary error function examples in chapter 6.

In the case of the error function integral in eq. (6.7),

$$y(z) = \mathrm{e}^{-z/\epsilon} \int_a^z \frac{\mathrm{e}^{f(t)/\epsilon}}{t}\, \mathrm{d}t \quad \text{where } f(t) = t, \tag{7.27}$$

the base asymptotic series is generated via the endpoint $t = z$. Stokes phenomena arises when the endpoint $t = z$ crosses the positive real axis, switching on the residue contribution at $t = 0$. This arises at $\operatorname{Im} f(z) = \operatorname{Im} f(0)$ and $\operatorname{Re} f(z) \geq \operatorname{Re} f(0)$.

The case of the complementary error function led to the analysis of (6.36)

$$\phi(z) = \int_z^\infty \mathrm{e}^{f(k)/\epsilon^2}\, \mathrm{d}k, \quad \text{where } f(k) = -k^2/2. \tag{7.28}$$

The base series, again from the endpoint, $k = z$, switches on the subdominant saddle contribution at $z = 0$ upon crossing the Stokes lines. The Stokes line is given by those values where $\operatorname{Im} f(z) = \operatorname{Im} f(0)$ and $\operatorname{Re} f(z) \geq \operatorname{Re} f(0)$, i.e. the imaginary axis.

Finally, the Airy integral in this chapter from (7.13),

$$\operatorname{Ai}(x) = \frac{\lambda^{1/3}}{2\pi\mathrm{i}} \int_{\infty \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}}^{\infty \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}} \mathrm{e}^{\lambda w(t)}\, \mathrm{d}t, \tag{7.29}$$

involves a base series generated from the saddle point $t = t_-$; Stokes phenomena occurs when this saddle switches on the subdominant saddle t_-. This occurs across the Stokes lines prescribed by those values of z where

$$\begin{aligned} \operatorname{Im} w(t_-, z) &= \operatorname{Im} w(t_+, z), \\ \operatorname{Re} w(t_-, z) &\geq \operatorname{Re} w(t_+, z). \end{aligned} \tag{7.30}$$

This occurs across $\operatorname{Arg} z = 2\pi/3$. A similar switching, with the secondary saddle switching off the primary at $\operatorname{Arg} z = -2\pi/3$.

7.3 EXPONENTIAL ASYMPTOTICS FOR THE AIRY EQUATION

The asymptotic analysis of the Airy equation we have presented in the previous section, done using the method of steepest descents, is a classical demonstration and can be found in numerous references (*e.g.* Bleistein and Handelsman 1986, Chap. 7). We now demonstrate how the Stokes phenomena can be predicted from analysis of the differential equation. To this end, let us return to (7.2) and re-label $f(z) = y(x)$.

> *Airy equation (ϵ version)*
>
> A re-scaling of the classic Airy equation yields the singularly perturbed version:
>
> $$\epsilon^2 y'' = xy, \tag{7.31a}$$
> $$y \to 0 \text{ as } x \to \infty, \tag{7.31b}$$
>
> to be studied in the limit $\epsilon \to 0$.

Our task is to explain how the details of the Stokes phenomenon (and Stokes line) can be explored via an analysis of the differential equation itself. The steps we follow are similar to the procedure first illustrated for the exponential integral in chapter 6, notably: (i) a naive study of the traditional asymptotic expansion of the solution; (ii) characterisation of the divergence of the series; (iii) an inner-region analysis in order to determine key constants; and (iv) optimal truncation and Stokes line smoothing.,

Again, we shall set the WKBJ ansatz $y \sim A(x)\mathrm{e}^{S(x)/\epsilon}$. Then at leading order as $\epsilon \to 0$, $(S')^2 = x$, so we select the branch of the square root corresponding to exponential decay of the solution. This gives

$$S(x) = -\frac{2}{3}x^{3/2}. \tag{7.32}$$

Above, we chosen the constant of integration so that the function $S = 0$ at the turning point, $x = 0$.

Let us now consider the full asymptotic expansion of the solution, as given by

$$y \sim A(x)\mathrm{e}^{-\frac{2}{3}x^{3/2}} \quad \text{where} \quad A(x) = \sum_{n=0}^{\infty} \epsilon^n A_n(x). \tag{7.33}$$

Substitution into (7.31a) now gives a differential equation for A:

$$\epsilon A'' + [2S'A' + S''A] = 0. \tag{7.34}$$

Setting the series expansion of A into the above equation give the leading-order solution

$$2S'A_0' + S''A_0 = 0 \Rightarrow A_0 = \frac{\text{const.}}{(S')^{1/2}} = \frac{1}{x^{1/4}}. \tag{7.35}$$

Above, we have chosen to set the leading constant in A_0 to be 1 for convenience (note this will differ from the scaling for the Airy function[3]).

For $n \geq 1$, we have at $\mathcal{O}(\epsilon^n)$:

$$A''_{n-1} + 2S'A'_n + S''A_n = 0. \tag{7.36}$$

In fact, this linear recurrence relation can be solved exactly, but for the moment, let us study the divergent properties of A_n as $n \to \infty$. Notice that the leading-order solution, A_0, in (7.35) is singular at $x = 0$.

[3] In order to precisely match the Airy Ai function, from (7.7), we would instead choose constant in (7.35) so that $A_0 = 1/(2\sqrt{\pi}x^{1/4})$. Alternatively all the results derived in this section can be multiplied by this factor.

Moreover, by the singular nature of the differential equation (on account of ϵ multiplying the highest derivative), derivation of A_1, A_2, and so forth will increase the severity of the singularity. In the limit $n \to \infty$, we may verify inductively or *a posteriori* that

$$A_n(x) \sim \frac{Q(x)\Gamma(n+\gamma)}{[\chi(x)]^{n+\gamma}}, \tag{7.37}$$

for functions $Q(x)$ and $\chi(x)$ to be determined and constant γ. Substitution of the factorial-over-power ansatz (7.37) into (7.36), and dividing by $Q\Gamma(n+\gamma)/\chi^{n+\gamma}$ gives

$$\begin{aligned}\left[(\chi')^2 - \left(\frac{2\chi' Q'}{Q} + \chi''\right)\frac{\Gamma(n+\gamma-1)}{\Gamma(n+\gamma)}\chi + \frac{Q''}{Q}\frac{\Gamma(n+\gamma-2)}{\Gamma(n+\gamma)}\chi^2\right] \\ 2S'\left(-\chi' + \frac{Q'}{Q}\frac{\Gamma(n+\gamma-1)}{\Gamma(n+\gamma)}\chi\right) + S''\frac{\Gamma(n+\gamma-1)}{\Gamma(n+\gamma)}\chi = 0.\end{aligned} \tag{7.38}$$

We now expand the ratio of the Gamma functions using Stirling's formula. This gives at leading order as $n \to \infty$, $\chi' = 2S'$, or

$$\chi(x) = 2S(x) = -\frac{4}{3}x^{3/2}, \tag{7.39}$$

where we have imposed the requirement that $\chi = 0$ at the singularity $x = 0$. Continuing to $\mathcal{O}(1/n)$, we have

$$Q(x) = \frac{\text{const.}}{(\chi')^{1/2}} = \frac{\Lambda}{x^{1/4}}. \tag{7.40}$$

where Λ is a constant to be determined.

7.3.1 Inner analysis

Our task now is to determine the constants Λ and γ in the factorial-over-power ansatz of (7.37). To begin, note that we have derived the following form for the late terms:

$$A_n \sim \frac{\Lambda}{x^{1/4}}\frac{\Gamma(n+\gamma)}{[-\frac{4}{3}x^{3/2}]^{n+\gamma}}, \qquad n \to \infty \tag{7.41}$$

and it remains to determine the values of γ and Λ. Firstly, the choice of γ ensures that the late-orders form above is consistent with the form of $A_0 = \mathcal{O}(x^{-1/4})$ in (7.35), which is the origin of the divergence. Thus setting $n = 0$ above we see that

$$\gamma = 0. \tag{7.42}$$

It now remains for us to determine the value of Λ by solving a recurrence relation near the inner region at $x = 0$. In the outer region, where x is away from $x = 0$, we have the asymptotic expansion of

$$A_{\text{out}}(x) \sim A_0(x) + \epsilon A_1(x) + \ldots + \epsilon^n \frac{Q(x)\Gamma(n+\gamma)}{[\chi(x)]^{n+\gamma}} + \ldots$$

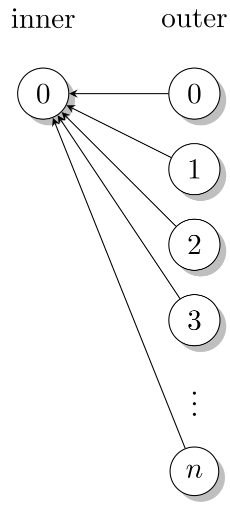

Figure 7.6: Illustration of the Van-Dyke matching procedure. The circles indicates the individual terms of an ϵ-series expansion. The number corresponds to the index, *i.e.* i for A_i. As x tends to the singularity, the outer expansion recombines to match the inner expansion. Note that for the case of the Airy equation, the inner problem consists of single non-ϵ-dependent problem. Thus there is only a single term of an ϵ expansion.

In the case of the Airy equation, the exact forms of $\chi = -\frac{4}{3}x^{3/2}$ and $Q(x) = \Lambda/x^{1/4}$ are known, and moreover, the functions are precisely their leading-order asymptotic limits as $x \to 0$. However, for the nonlinear examples later presented in this book, more complicated forms will be involved. As $x \to 0$, the outer expansion breaks down, with each term of the approximation merging to be of the same order. This is shown schematically in fig. 7.6. For instance, the nth term will be of the same order[4] as the leading term if $x = \mathcal{O}(\epsilon^{2/3})$, and this indicates the boundary layer size.

[4] Note that to balance the nth term with the 0th term, we have $\epsilon^n/[x^{1/4}x^{3/2n}] = \mathcal{O}(x^{1/4})$.

The solution in the inner region is expected to be on the same order as $A_0(x) = \mathcal{O}(x^{1/4})$. Hence this motivates the re-scalings of

$$A(x) = \frac{G(\zeta)}{x^{1/4}} \quad \text{and} \quad \zeta = -\frac{4}{3}\frac{x^{3/2}}{\epsilon}. \tag{7.43}$$

The extra factor of $-4/3$ in the inner coordinate, ζ, is essentially so that the late-term expression (7.41) yields exact powers of ζ.

The inner problem becomes

$$\frac{\mathrm{d}^2G}{\mathrm{d}\zeta^2} + \frac{\mathrm{d}G}{\mathrm{d}\zeta} + \frac{5G}{36\zeta^2} = 0, \tag{7.44}$$

subject to $G \to 1$ as $z \to \infty$. Based on the form of the solution in the outer region, we expect that G is expanded as a series in inverse powers of ζ as $\zeta \to \infty$. Thus we substitute

$$G(\zeta) = \sum_{n=0}^{\infty} \frac{G_n}{\zeta^n} \tag{7.45}$$

to obtain a recurrence relation:

$$G_{n+1} = \left(n + \frac{5}{36}\frac{1}{n+1}\right) G_n, \qquad n \geq 1 \tag{7.46}$$

and this can be solved explicitly[5] to yield the values of

[5] Mathematica's RSolve can do this.

$$G_n = \frac{1}{2\pi}\frac{\Gamma(n+1/6)\Gamma(n+5/6)}{\Gamma(n+1)}, \tag{7.47}$$

where note $G_0 = 1$. Indeed, the expansion of the leading-order solution also diverges as it approaches the outer region.

In general and for more complicated problems, we would not expect to be able to solve the recurrence relation exactly for all n. However, for such problems, it is typically only necessary to estimate the divergence of the series (7.45) as $n \to \infty$. A direct numerical calculation yields fig. 7.7.

In our case, we can verify by Stirling's approximation[6] that

[6] In the limit that $|z| \to \infty$, Stirling's approximation is that $\Gamma(z) \sim \sqrt{2\pi}\exp(z)\exp[(z-1/2)\log z]$.

$$G_n \sim \frac{\Gamma(n)}{2\pi} \quad \text{as } n \to \infty. \tag{7.48}$$

The comparatively simplistic nature of the Airy equation makes matching between inner and outer solutions straightforward. Recalling that $\chi = \epsilon\zeta$, the nth term of the outer approximation is

$$A_{\text{outer}} \Longrightarrow \epsilon^n A_n(x) \sim \frac{1}{x^{1/4}}\frac{\Lambda\Gamma(n)}{\zeta^n} \qquad \text{as } x \to 0. \tag{7.49}$$

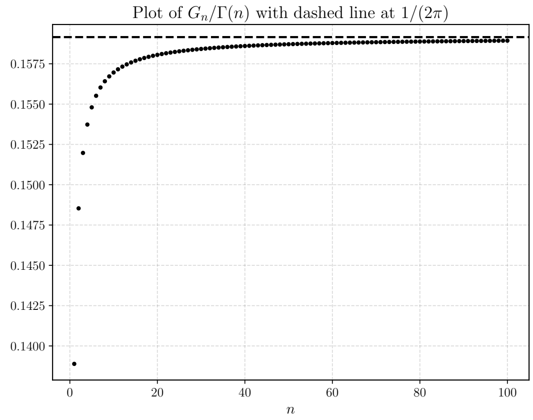

Figure 7.7: Numerical solution of the recurrence relation (7.46) showing convergence to $G_n \sim 1/(2\pi)$ as $n \to \infty$.

This is then matched to the nth term of the inner solution. From (7.48), we have

$$A_{\text{inner}} \Longrightarrow \frac{G_n}{x^{1/4}\zeta^n} \sim \frac{\Gamma(n)}{2\pi x^{1/4}\zeta^n}, \tag{7.50}$$

therefore

$$\Lambda = \frac{1}{2\pi}. \tag{7.51}$$

The determination of Λ above completes our characterisation of the divergent form (7.37).

7.3.2 Optimal truncation and Stokes smoothing

The Stokes smoothing procedure proceeds similarly to the analogous procedure in chapter 6. We first truncate the expansion,

$$A(x) = \sum_{n=0}^{N-1} \epsilon^n A_n + R_N(x), \tag{7.52}$$

and obtain an equation for the remainder, $R_N(x)$, which is forced by the divergence of the base asymptotic expansion. If N is chosen optimally, then R_N is exponentially small, and its form is sought as x varies across the Stokes line. Using a similar procedure to that presented in chapter 6, we find that across the Stokes lines (proceeding in the anti-clockwise direction), the remainder $R_N(x)$ incurs a jump given by

$$R_N(x) \sim \left[\frac{2\pi\mathrm{i}}{\epsilon^\gamma}\right] Q\mathrm{e}^{-\chi/\epsilon} = \mathrm{i}\frac{\mathrm{e}^{4/3x^{4/3}/\epsilon}}{x^{1/4}}, \tag{7.53}$$

where χ and Q are respectively given by (7.39) and (7.40), with further components of $\gamma = 0$ and $\Lambda = 1/(2\pi)$.

The criterion for the location of Stokes lines had been earlier derived for the case of the exponential integral and complementary error functions using the Stokes line smoothing procedures (where we would have noted the common factor of $\mathrm{e}^{\chi/\epsilon - \|\chi|/\epsilon}$ (see the exercises at the end of this section). Therefore, for this case, they are given in the same fashion as in (6.26):

Stokes lines for the Airy equation (algebraic base series version)

The Stokes lines for the Airy equation, with solution written as $y = A\mathrm{e}^{-2/3x^{3/2}/\epsilon}$, are given by the set of points, x, where

$$\operatorname{Im}\chi(x) = 0 \quad \text{and} \quad \operatorname{Re}\chi(x) \geq 0, \tag{7.54}$$

where $\chi = -\frac{4}{3}x^{3/2}$. Across such lines (curves), the base series, $A \sim A_0 + \epsilon A_1 + \epsilon^2 A_2 + \ldots$ with $A_0 = 1/x^{1/4}$ switches-on the exponential $(\mathrm{i}/x^{1/4})\mathrm{e}^{-\chi/\epsilon}$.

Returning to the solution in terms of y via (7.33) and multiplying this remainder with the base exponential scaling of $\mathrm{e}^{-2/3x^{3/2}/\epsilon}$, then we have that across the Stokes line, the following switching occurs:

$$\frac{1}{x^{1/4}}\mathrm{e}^{-2/3x^{3/2}/\epsilon} \xrightarrow{\theta=2\pi/3} \frac{1}{x^{1/4}}\mathrm{e}^{-2/3x^{3/2}/\epsilon} + \mathrm{i}\frac{1}{x^{1/4}}\mathrm{e}^{2/3x^{3/2}/\epsilon}. \tag{7.55}$$

We can check its value on the negative real axis. Taking $x^{3/2} = -\mathrm{i}|x|^{3/2}$ (the branch cut is taken along $\operatorname{Arg}(x) = 2\pi$), we have

$$\frac{2}{|x|^{1/4}}\cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right). \tag{7.56}$$

This needs to be checked!

Remembering that our solutions in this section should be multipled by $1/(2\sqrt{\pi})$ in order to match the scaling on the Ai function, then we can now compare the above to (7.33).

The exponential asymptotics is illustrated in fig. 7.8, showing the smooth switching-on of the exponentially-small term across $\theta = 2\pi/3$.

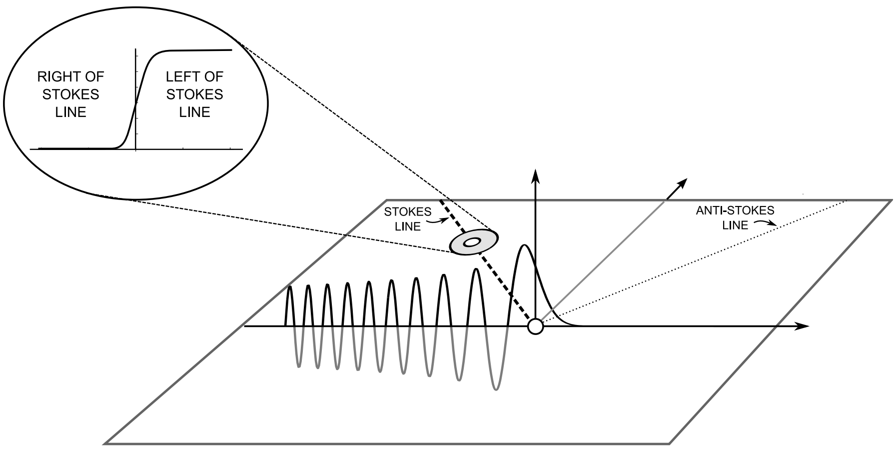

Figure 7.8: The Ai(x) function (shown solid) is analytically continued into the complex x-plane. At $\operatorname{Arg} x = 2\pi/3$, the transition across the Stokes line (dashed) causes an exponentially small term to switch on in a smooth manner.

Further notes

The reader will recall that, related to (7.32), we had removed the leading exponential from the original $y(x)$ form of the Airy equation; this was convenience, allowing the base series expansion for $A(z)$, to proceed in a purely algebraic series in powers of ϵ. In this case, the Stokes line criterion leading to (7.54) effectively compares the subdominant exponential, $\sim \mathrm{e}^{-\chi/\epsilon}$ with a base series with leading exponential e^0. If we instead label

$$\chi_1 = \frac{2}{3}x^{3/2} \quad \text{and} \quad \chi_2 = -\frac{2}{3}x^{3/2}, \tag{7.57}$$

and note that two asymptotic expansions for y being considered as proportional to $\mathrm{e}^{-\chi_1/\epsilon}$ and $\mathrm{e}^{-\chi/\epsilon}$, then we can develop the following alternative Stokes line condition:

Stokes lines for the Airy equation (exponential base series version)

The Stokes lines for the Airy equation are given by where

$$\operatorname{Im}\chi_2(x) = \operatorname{Im}\chi_1(x) \quad \text{and} \quad \operatorname{Re}\chi_2(x) \geq \operatorname{Re}\chi_1(x), \tag{7.58}$$

where χ_1 and χ_2 are defined as above. Across such curves, the base series, with $y \sim x^{-1/4}\mathrm{e}^{-\chi_1/\epsilon}$ switches on the exponential $\sim \mathrm{i}x^{-1/4}\mathrm{e}^{-\chi_2/\epsilon}$.

The relationship between the Stokes line condition above and the steepest descent criterion in (7.30) should now be understood.

7.4 EXERCISES

1. By substituting the WKBJ ansatz $f \sim A(z)\mathrm{e}^{-\chi(z)/\epsilon}$ into the Airy equation (7.2), develop the two linearly independent solutions given by (7.6).
2. By performing optimal truncation and Stokes line smoothing applied to the ODE (7.34), demonstrate that the remainder satisfies (7.53).