THE EXPONENTIAL INTEGRAL AND COMPLEMENTARY ERROR FUNCTION

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6.1 THE EXPONENTIAL INTEGRAL

Differential equation for the exponential integral function

One of the simplest toy models for exponential asymptotics is the study of the first-order linear differential equation,

$$\epsilon \frac{dy}{dz} + y = \frac{\epsilon}{z} \tag{6.1a}$$

$$y \to 0 \quad \text{as } z \to -\infty - b\mathrm{i}, \tag{6.1b}$$

for $y = y(z)$ where $z \in \mathbb{C}$ and $b > 0$.

The function y defined as above is connected to the special function Ei, known as the exponential integral. Notice that we have chosen the boundary condition in (6.1b) so that y decays to zero as z tends to $-\infty$ in the lower-half complex plane. Our goal is to study the asymptotics of (6.1) as $\epsilon \to 0$.

Although the equation can be solved by integration, we pretend to be ignorant of this fact. Let us seek a regular series expansion in powers of ϵ. Setting

$$y(z) = \sum_{n=0}^{\infty} \epsilon^n y_n(z), \tag{6.2}$$

this gives

$$y_0(z) = 0, \tag{6.3a}$$

$$y_n(z) = -y'_{n-1}(z), \quad n > 1, \tag{6.3b}$$

and thus solving at each order yields

$$y(z) = 0 + \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \ldots = \sum_{n=1}^{\infty} \frac{\epsilon^n (n-1)!}{z^n} = \sum_{n=1}^{\infty} \frac{\epsilon^n \Gamma(n)}{z^n}, \tag{6.4}$$

and we have exchanged the factorial representation $(n-1)!$ for a more general Gamma function $\Gamma(n)$ for later generality. For any fixed value of ϵ and z, note that the above series must diverge since the general term $\epsilon^n \Gamma(n)/z^n$ is unbounded as $n \to \infty$.

In the above procedure, we have anticipated the form of the terms of the expansion, that is $\epsilon^n \Gamma(n)/z^n$, based on the ease of solving the relationship for the $\mathcal{O}(\epsilon^n)$ term in (6.3b). However, divergence in the limit $n \to \infty$ can be understood from a much more generic argument that does not depend on our ability to explicitly solve the scheme at each order. The following argument will form the basis of our approaches for the nonlinear equations to follow.

Notice that the first approximation yields

$$y_1(z) = \frac{1}{z}, \tag{6.5}$$

which is singular at the origin. Since every subsequent order involves differentiating the previous order via (6.3b), then every subsequent order must produce a new multiplicative factor (of the denominator), which increases the power of the previous order. Thus $y_2 = -1/z^2$, $y_3 = 2/z^3$, and so forth. The result diverges in the form of a *factorial over power.*

Late terms of the exponential integral function

As a more general representation, we might consider describing the late-orders divergence as

$$y_n \sim \frac{\Gamma(n+\gamma)}{[\chi(z)]^{n+\gamma}} \qquad \text{with } \gamma = 0 \text{ and } \chi(z) = z, \tag{6.6}$$

valid in the limit $n \to \infty$. The particular case of the exponential integral, the above expression is an equality and applicable at each order; for this case, $\gamma = 0$ and $\chi = z$.

Let us consider how well the asymptotic approximation (6.4) approximates the exact solution. We consider the following numerical experiment illustrated in Figure 6.1. The differential equation (6.1a) is solved by integrating from a point $z = A$ to a point $z = B$ along the path illustrated in the figure. Since the point A is chosen to lie at a finite point rather than $-\infty$ (see the boundary condition (6.1b)), we shall choose as the initial condition $y(A) = \epsilon/A$ so that the solution initially matches the leading-order asymptotic approximation. Once the point $z = B$ is reached, we continue solving the differential equation along the second segment and towards the point $z = C$.

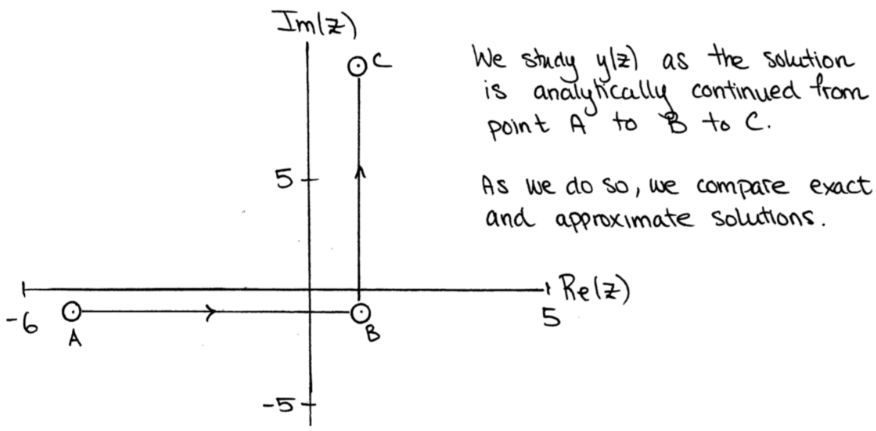

Figure 6.1: Analytic continuation experiment for the exponential integral

The result of this numerical solution procedure for the case of $\epsilon = 0.3$ is shown in Figure 6.2, where the real part of the solution, $\text{Re}(y)$, is plotted as a function of the arclength along the segments AB and BC shown in Figure 6.1. As we can observe, there is excellent agreement between the exact solution and $y \sim \epsilon/z$ along the segment AB. However, as the solution is then analytically continued along the segment BC, we observe the peculiar phenomenon whereby the exact solution exhibits oscillatory behaviour, and yet no term from the expansion (6.4) contains an oscillatory component. In fact, a more thorough investigation would have revealed that the magnitude of the oscillations along the segment BC is exponentially small and on the order of $\mathrm{e}^{-\text{const.}/\epsilon}$. Let us now investigate this peculiarity.

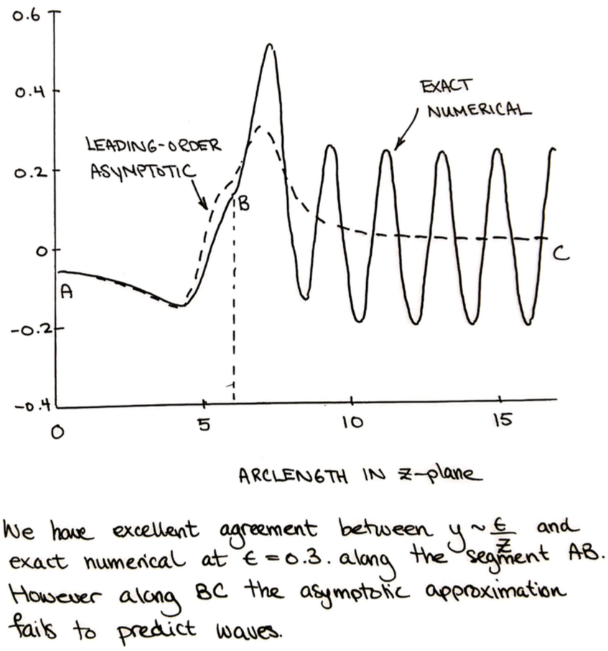

Figure 6.2

6.1.1 *Analysis by the method of steepest descents*

Consider integrating (6.1a). This gives

$$y(z) = \mathrm{e}^{-z/\epsilon} \int_a^z \frac{\mathrm{e}^{t/\epsilon}}{t} \, \mathrm{d}t, \tag{6.7}$$

where we shall then take $a \to -\infty - b\mathrm{i}$, .

Our aim is to apply the method of steepest descents (chapter 5) in order to develop the asymptotic approximation to (6.7). Conceptually, it is easier to consider the point $t = a$ as a finite point in the lower half-z-plane; thus we deform the initial contour between $t = a$ to $t = z$. By the method of steepest descents, paths of steepest descent or ascent lie on the equal-phase contours [cf. (5.6) with $\varphi = t$] given by $\mathrm{Im}(t) = \text{const.}$ and hence they are horizontal lines in the t-plane. As $\epsilon \to 0$, the integrand is either exponentially large or exponentially small dependent on $\mathrm{Re}(t)$.

Suppose firstly that $\mathrm{Im}(z) < 0$ and so the initial contour of integration lies entirely in the lower half-plane. In this case, the deformation procedure is shown in Figure 6.3. In the limit $\epsilon \to 0$, the dominant contributions to the integral are from the endpoints. In essence this justifies an approximation using integration by parts (section 4.1):

$$y = \mathrm{e}^{-z/\epsilon}\left[\frac{\epsilon \mathrm{e}^{t/\epsilon}}{t}\bigg|_a^z + \epsilon \int_a^z \frac{\mathrm{e}^{t/\epsilon}}{t^2}\,\mathrm{d}t\right] \sim \frac{\epsilon}{z} + \epsilon \mathrm{e}^{-z/\epsilon}\int_a^z \frac{\mathrm{e}^{t/\epsilon}}{t^2}\,\mathrm{d}t, \qquad (6.8)$$

where we take $a \to -\infty + b\mathrm{i}$. Subsequent application of integration by parts yields further terms of the asymptotic expansion (6.4). In Exercise 6.4, we can also apply the method of steepest descents directly to produce the above expansion.

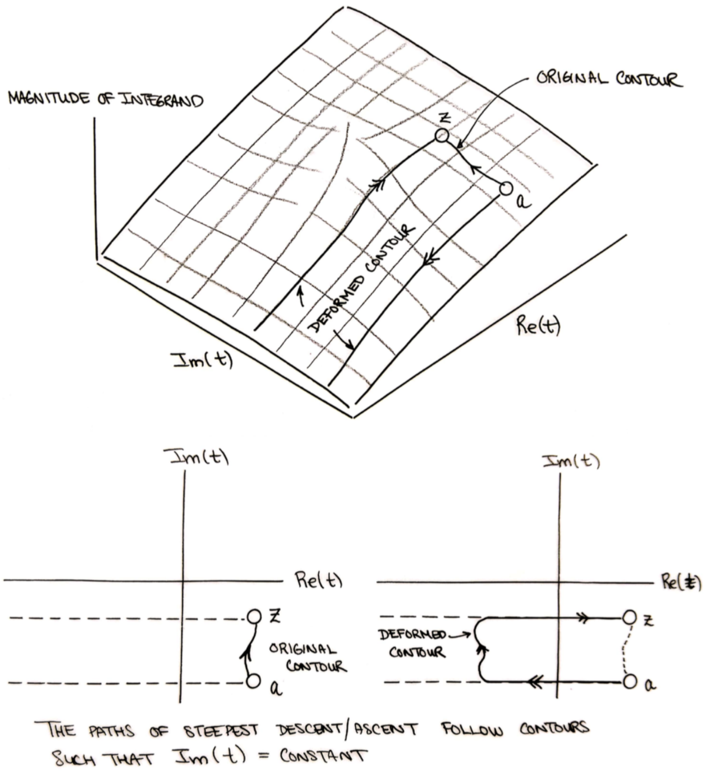

Figure 6.3

Consider now the approximation of 6.7 when the initial contour of integration from $t = a$ to $t = z$ is such that $\mathrm{Im}(z) > 0$ in $\mathrm{Re}(t) > 0$ plane as shown in fig. 6.4. Now the deformation must include the residue contribution in addition to the previously derived asymptotic expansion,

$$y = \left[\frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \dots\right] + \mathrm{e}^{-z/\epsilon}\oint_{z=0} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t = \left[\frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \dots\right] + 2\pi \mathrm{i}\mathrm{e}^{-z/\epsilon}. \qquad (6.9)$$

This is the *Stokes phenomenon*—the sudden switching on (or off) of exponentials across *Stokes lines*, here given by $\mathrm{Re}\, z \geq 0$. Notice that once $\mathrm{Re}(z) < 0$, then the exponentially small contribution is then exponentially large.

Many authors do not provide a strict definition of Stokes phenomena, preferring to establish its meaning via practical demonstrations and by

We sketch the calculations required when applying the method of steepest descents, instead of the direct integration by parts which we have used in (6.8). Consider the approximation of the integral component of eq. (6.7),

$$I = \int_{-\infty}^{z} \frac{\mathrm{e}^{t/\epsilon}}{t}\,\mathrm{d}t.$$

From the endpoint at $t = z$, the steepest descent path follows the line constant phase of the exponential, where $\mathrm{Im}(t)$ is constant. According to the steepest descent methodology, we thus set $t = z - s$ for $s \in [0, \infty)$. Consequently, the integral is approximated about its endpoint, $s = 0$, with

$$I \sim -\int_0^\delta \frac{\mathrm{e}^{(z-s)/\epsilon}}{z - s}\,\mathrm{d}s,$$

and $\delta \ll 1$. Expanding the denominator near $s = 0$ yields

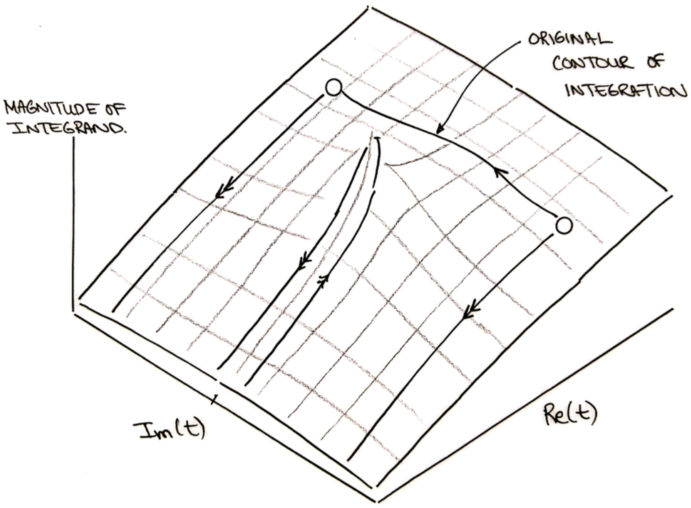

Figure 6.4

example. In some cases, it is defined more rigorously for specific limited systems (*e.g.* systems of linear differential equations). Such definitions are typically based on describing how the solutions of a linear differential equation are described by different linear combinations of asymptotic approximations, valid in different sectors of the complex plane. The Stokes phenomenon then corresponds to when the coefficients of such combinations may vary from sector to sector (cf. Erdélyi 1965). We prefer the former approach, and present only an informal definition.

Definition 6.1 (Stokes phenomenon and Stokes lines)
Given an asymptotic expansion, say

$$y(z) \sim y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \dots, \tag{6.10}$$

valid in the Poincaré sense for $\epsilon > 0$ and $\epsilon \to 0$, in some region of the complex plane, $z \in S \subset \mathbb{C}$, the *Stokes phenomenon* corresponds to the 'switching-on' of exponentially-small terms, beyond-all-orders of the above expansion, across some curve in the complex plane. That is, across such curves, the above asymptotic expansion is modified to, *e.g.*

$$y(z) \sim \Big[y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \dots\Big] + A(z)\mathrm{e}^{-\chi(z)/\epsilon}. \tag{6.11}$$

The curves across which the Stokes phenomena occurs is called the *Stokes line.*

There is much that can be said about the insufficiency of the above definition. It should really be extended to cover much more general asymptotic series than (6.10); there is a lot more to say about the switching (6.11) as well. But the definition seems adequate for now; as our understanding of the theory deepens, we may attempt a more all-encompassing definition.

6.1.2 Optimal truncation and Stokes line smoothing

The main question is how to predict the Stokes phenomenon without access to the closed-form formula representation (6.7). After all, in more general linear (or certainly nonlinear) formulations, the solution may not be expressible in terms of an integral form where approximation methodologies like the method of steepest descent can be applied. In particular, we seek a more general procedure that is applicable to differential equations. Returning to the differential equation form of the exponential integral in (6.1a), we wish to establish a link between divergence of the underlying asymptotic expansion and the emergence of exponentially-small terms.

We truncate the series (6.4) after N terms,

$$y = \sum_{n=0}^{N-1} \epsilon^n y_n + R_N. \tag{6.12}$$

Substitution of the truncated expansion into (6.1a) gives

$$\mathcal{L}R_N = -\epsilon^N y'_{N-1} = \epsilon^N y_N, \tag{6.13}$$

where we have defined the linear operator $\mathcal{L}$ such that

$$\mathcal{L}R_n \coloneqq \epsilon R'_N + R_N. \tag{6.14}$$

and hence we have a linear equation for the remainder R_N. For general choices of N, the right hand-side indicates that the remainder is $\mathcal{O}(\epsilon^N)$ in size and algebraically small as expected.

However, it is possible to truncate the asymptotic approximation optimally and minimise the remainder. A rule-of-thumb for this *optimal truncation point* is that it should be performed at the term that produces the smallest contribution to the sum—or equivalently, the point where adjacent terms of the expansion are equal in size[1].

[1]Our prescription for optimal truncation is generally treated as a rule-of-thumb or asymptotics folklore; in specific cases it can be justified *a priori* Berry and Howls 1990, Berry 1991b, Costin and Kruskal 1999. In many other works, however, optimal truncation is justified *a posteriori*.

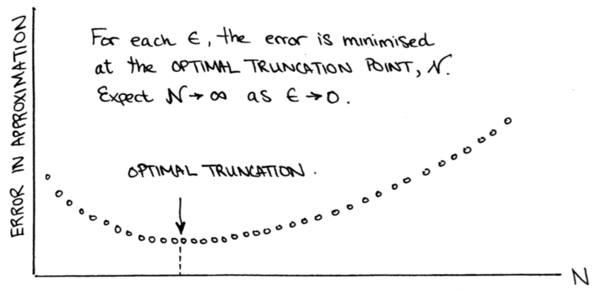

Figure 6.5

In order to facilitate comparison with later examples, we will continue to represent the terms of the asymptotic expansion in the form of (6.6), or

$$y_n = \frac{\epsilon^n \Gamma(n+\gamma)}{[\chi(z)]^{n+\gamma}}. \tag{6.15}$$

For the case of the exponential integral example, $\gamma = 0$ and $\chi = z$.

Then, using the rule-of-thumb that the optimal truncation point is given at the point where adjacent terms are approximately equal in size (for a fixed value of z), we obtain

$$\left|\frac{\epsilon^{N+1}y_{N+1}}{\epsilon^N y_N}\right| = \left|\frac{\epsilon\Gamma(N+\gamma+1)\chi^{N+\gamma}}{\Gamma(N+\gamma)[\chi(z)]^{N+\gamma+1}}\right| = \frac{\epsilon N}{|\chi(z)|} \sim 1,$$

imposed in the limit $\epsilon \to 0$. Thus we shall choose to truncate at

$$N = \frac{|\chi(z)|}{\epsilon} + \rho = \frac{|z|}{\epsilon} + \rho. \tag{6.16}$$

The constant $\rho \in [0, 1)$ is introduced so that N is an integer (the ceiling of $|\chi|/\epsilon$). Crucially, notice that as $\epsilon \to 0$ the truncation point $N \to \infty$, and this relationship is intrinsically linked to the asymptotic nature of the approximations.

Moreover, the above commentary on the optimal truncation point partly establishes the important correspondence between the remainder R_N and divergent terms found in (6.13). Since the remainder obeys,

$$\mathcal{L}R_N = -\epsilon^N y_N, \tag{6.17}$$

then in the limit $\epsilon \to 0$ and with $N \to \infty$ the behaviour of the remainder is hence governed by the divergence of y_N. This phenomena is essentially why many authors describe beyond-all-orders procedures as *'decoding the divergence'*—that is, despite the apparently nonsensical nature of a divergent series, the divergent tail of a singular asymptotic expansion nevertheless contains key information that can be extracted.

Let us now return to the study of the equation for the remainder above. Note that the homogeneous solution of $\mathcal{L}R_N = 0$ is given by

$$R_N(z) \sim A\mathrm{e}^{-\chi(z)/\epsilon} = A\mathrm{e}^{-z/\epsilon}, \tag{6.18}$$

where A is an arbitrary constant[2].

In order to solve the equation for the remainder, (6.13), it is prudent to attempt the solution form

$$R_N(z) = \mathcal{S}(z)\mathrm{e}^{-\chi(z)/\epsilon}, \tag{6.19}$$

where $\mathcal{S}$ is the Stokes smoothing function, and we expect for it to smoothly increase from one constant to another across Stokes lines (which we expect to detect). Differentiating the above, we have

$$R_N' = \left(-\frac{\chi'}{\epsilon}\mathcal{S} + \mathcal{S}'\right)\mathrm{e}^{-\chi/\epsilon}. \tag{6.20}$$

When we substitute the above two expressions for the remainder, (6.19) and (6.20) into (6.13), the terms on the left hand-side of (6.13) scaling with $\mathcal{S}$ sum to zero by design of the homogeneous part, $\mathrm{e}^{-\chi/\epsilon}$. This leaves

$$\epsilon\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}\mathrm{e}^{-\chi/\epsilon} \sim \epsilon^N \frac{\Gamma(N+\gamma)}{\chi^{N+\gamma}}. \tag{6.21}$$

[2] It is certainly not a coincidence that the χ that appears in the solution of the homogeneous solution above is the same χ that has been used to characterise the divergence (6.15). However, the significance or generality of this connection might be unclear at this stage

Since optimal truncation occurs with $N \to \infty$, we shall expand the Gamma function using Stirling's approximation. We substitute $N = |\chi|/\epsilon + \rho$ into the Stirling approximation to yield

$$\Gamma(N+\gamma) \sim \sqrt{2\pi}(N+\gamma)^{N+\gamma-1/2}\mathrm{e}^{-N-\gamma}$$
$$\sim \sqrt{2\pi}\left(\frac{|\chi|}{\epsilon}\right)^{N+\gamma-1/2}\left(1+\frac{(\rho+\gamma)\epsilon}{|\chi|}\right)^{N+\gamma-1/2}\mathrm{e}^{-N-\gamma}. \tag{6.22}$$

The round bracketed factor is expanded to $\mathrm{e}^{\rho+\gamma} + \mathcal{O}(\epsilon)$. This then yields a result that we will often use in the forthcoming derivations:

Stirling's formula manipulation

If $N = |\chi|/\epsilon + \rho$, we can expand the gamma function as

$$\Gamma(N+\gamma) \sim \sqrt{2\pi}\left(\frac{|\chi|}{\epsilon}\right)^{N+\gamma-1/2}\mathrm{e}^{-\frac{|\chi|}{\epsilon}}, \tag{6.23}$$

in the limit $\epsilon \to 0$.

Thus returning to (6.21) and using (6.23), we have

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z} \sim \left[\sqrt{\frac{2\pi}{|\chi|}}\frac{1}{\epsilon^{\gamma+1/2}}\right]\frac{\mathrm{e}^{\chi/\epsilon}\mathrm{e}^{-|\chi|/\epsilon}}{(\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}\,\chi})^{N+\gamma}}. \tag{6.24}$$

Again, remember that for the case of the exponential integral, we have $\gamma = 0$ and $\chi = z$.

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z} \sim \left[\sqrt{\frac{2\pi}{|z|}}\frac{1}{\epsilon^{1/2}}\right]\frac{\mathrm{e}^{z/\epsilon}\mathrm{e}^{-|z|/\epsilon}}{(\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}\,z})^{N}}. \tag{6.25}$$

The key feature to notice is that if the rate-of-change of S is measured haphazardly, then in general $\chi - |\chi| < 0$ and thus $\mathrm{e}^{(\chi-|\chi|)/\epsilon}$ is exponentially small. However, if z is chosen such that

$$\mathrm{Im}\,\chi = \mathrm{Im}\,z = 0 \qquad \text{and} \qquad \mathrm{Re}\,\chi = \mathrm{Re}\,z \geq 0, \tag{6.26}$$

then the right hand-side is in fact algebraically small in ϵ. This process has indeed revealed the location of the Stones line—i.e. those points $z \in \mathbb{C}$ where (6.26) is satisfied! Thus the Stokes line is the positive real axis, as established using the direct analysis of the integral equation.

6.1.3 *Integration about the Stokes line*

Our goal is to now examine (6.24) and obtain the change in $\mathcal{S}(z)$ about the Stokes line, which was determined to be along $\mathrm{Re}\,z > 0$. Recall that the optimal truncation criteria (6.16) was $N = |z|/\epsilon + \rho$. Thus for a given ϵ, and with N fixed in the prescribed fashion, we wish to consider what happens to the remainder as the argument of z increases from negative values to positive values. Thus we set $z = re^{i\vartheta}$ and consider the behaviour of the solution near $\vartheta = 0$.

For simplicity, we shall perform the procedure as applied to the equation for $\frac{d\mathcal{S}}{dz}$ in (6.25) having substituted $\gamma = 0$ and $\chi = z$. It will be useful for the reader to duplicate these steps for the more general representation in (6.24).[3]

[3] See exercises in section 6.4

Differentiation now follows

$$\frac{d}{dz} = \frac{d\vartheta}{dz}\frac{d}{d\chi} = \left(\frac{1}{ire^{i\vartheta}}\right)\frac{d}{d\vartheta}.$$

Substitution into (6.24) yields

$$\frac{d\mathcal{S}}{d\vartheta} \sim \left[i\frac{\sqrt{2\pi r}}{\epsilon^{1/2}}\right]\frac{e^{z/\epsilon}e^{-|z|/\epsilon}}{(e^{i\vartheta})^{N-1}}. \tag{6.27}$$

We now write the right hand-side in terms of $z = re^{i\vartheta}$, giving

$$\frac{d\mathcal{S}}{d\vartheta} \sim \left[i\frac{\sqrt{2\pi r}}{\epsilon^{1/2}}\right]\exp\Big(\frac{r}{\epsilon}(e^{i\vartheta} - 1 - i\vartheta) - i\vartheta(\rho - 1)\Big). \tag{6.28}$$

Thus for ϑ small,

$$\frac{d\mathcal{S}}{d\vartheta} \sim \left[i\frac{\sqrt{2\pi r}}{\epsilon^{1/2}}\right]\exp\left(\frac{-r\vartheta^2/2 + O(\vartheta^3)}{\epsilon} + \mathcal{O}(\vartheta)\right). \tag{6.29}$$

Crucially, notice that the above expression signifies that the amplitude factor $\mathcal{S}$ changes rapidly (as a Gaussian) in a boundary layer around $\vartheta = 0$. The critical scaling is given by $\vartheta = \epsilon^{1/2}\overline{\vartheta}$. Changing now to the re-scaled inner coordinate $\overline{\vartheta}$, we have

$$\frac{d\mathcal{S}}{d\overline{\vartheta}} \sim \left[i\sqrt{2\pi r}\right]\exp\Big(-r\overline{\vartheta}^2/2 + \mathcal{O}(\epsilon^{1/2})\Big). \tag{6.30}$$

It is useful to introduce the notation of the Error function, $\mathrm{erf}(z)$[4]. Using the fact that $\mathcal{S} \to 0$ as $\overline{\vartheta} \to -\infty$, we may then integrate the above expression about the boundary layer to obtain

[4] The Error function is defined by $\mathrm{erf}(z) = \frac{2}{\pi}\int_0^z e^{-t^2}\, dt$.

$$\mathcal{S} \sim i\sqrt{2\pi r}\int_{-\infty}^{\overline{\vartheta}} \exp\big(-rs^2/2\big)\, ds = \pi i\left[1 + \mathrm{erf}\left(\sqrt{\frac{r}{2}}\overline{\vartheta}\right)\right]. \tag{6.31}$$

The contribution across the Stokes line is derived by taking $\overline{\vartheta} \to \infty$, as sketched in fig. 6.6.

Since $\mathrm{erf}(u) \to 1$ as $u \to \infty$, this gives a total jump of

$$\mathcal{S}(\overline{\vartheta} \to \infty) - \mathcal{S}(\overline{\vartheta} \to -\infty) \equiv [S] \sim 2\pi i. \tag{6.32}$$

Hence across Stokes lines, we switch on the remainder,

$$R_N \sim 2\pi i e^{-z/\epsilon} \tag{6.33}$$

At last, we have obtained the same result [cf. (6.9)] that had been derived earlier through integral approximation procedures.

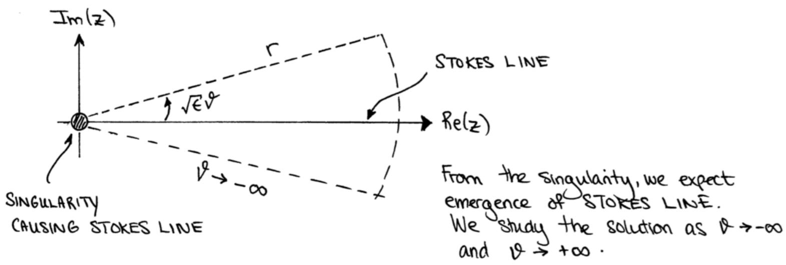

Figure 6.6

6.2 THE COMPLEMENTARY ERROR FUNCTION

The complementary error function presents an alternative introduction to the principles of exponential asymptotics. In contrast to our previous example of the exponential, the example from this section is somewhat different for a pair of reasons: first, its integral representation exhibits the Stokes phenomenon through the switching-on and -off of a saddle point of the integrand rather than a pole; and (ii) the leading-order asymptotic expansion is no longer purely algebraic and occurring in powers of ϵ. Nevertheless, as we shall see, the principles are largely the same.

Let us define $\phi = \phi(z)$ according to:

> *Complementary error function problem*
>
> Consider $\phi = \phi(z)$ defined by
>
> $$\epsilon^2\phi'' + z\phi' = 0, \tag{6.34a}$$
>
> with primes for differentiation in z, and with boundary conditions
>
> $$\phi(0) = \epsilon\sqrt{\frac{\pi}{2}}, \tag{6.34b}$$
>
> $$\phi(\infty) = 0, \tag{6.34c}$$
>
> considered in the limit $\epsilon \to 0$.

For ease of explanation, we shall first consider $z \in \mathbb{R}$ and with the second boundary condition above tending to infinity along the positive real axis.

By direct integration, the solution of the above problem can also be written as

$$\phi(z) = \int_z^\infty e^{-k^2/(2\epsilon^2)}\, dk = \epsilon\sqrt{\frac{\pi}{2}}\, \text{erfc}\left(\frac{z}{\sqrt{2}\epsilon}\right), \tag{6.35}$$

where erfc(z) is the complementary error function[5].

As with the exponential integral, the two equivalent representations of ϕ in differential-equation form of (6.34) or integral-equation form (6.35) provides an opportunity for multiple approaches to analysing the

[5] The complementary error function is defined by (see e.g. Abramowitz and Stegun 1983)

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-k^2}\, dk.$$

underlying asymptotics. The analysis of the integral equation is most straightforward so we begin there.

6.2.1 *Analysis by the method of steepest descents*

We write the integral (6.35) in the Laplace-integral form of

$$\phi(z) = \int_z^\infty \mathrm{e}^{f(k)/\epsilon^2}\, \mathrm{d}k, \quad \text{where } f(k) = -k^2/2. \tag{6.36}$$

Notice that there is a saddle point where $f'(k) = 0$, i.e. where $k = 0$. In order to develop the asymptotic expansion of the integral (6.36) as $\epsilon \to 0$, we deform the contour of integration, which originally lies from $k = z$ to $k = \infty$, along paths of steepest descent.

Note that the paths of constant phase are given $\mathrm{Im}(z^2)$ constant and thus the hyperbolae $\mathrm{Re}(k)^2 - \mathrm{Im}(k)^2 = \text{const.}$

Consider firstly the situation where the initial point, z, lies within the sector $-\pi/2 < \theta < \pi/2$, where $\theta = \arg(z)$. Here, the initial point is connected to a path of steepest descent that already tends to $+\infty$ and is given by the constant phase path $\mathrm{Im}[f(k)] = \mathrm{Im}[f(z)]$. Then the integration should proceed as shown in fig. 6.7.

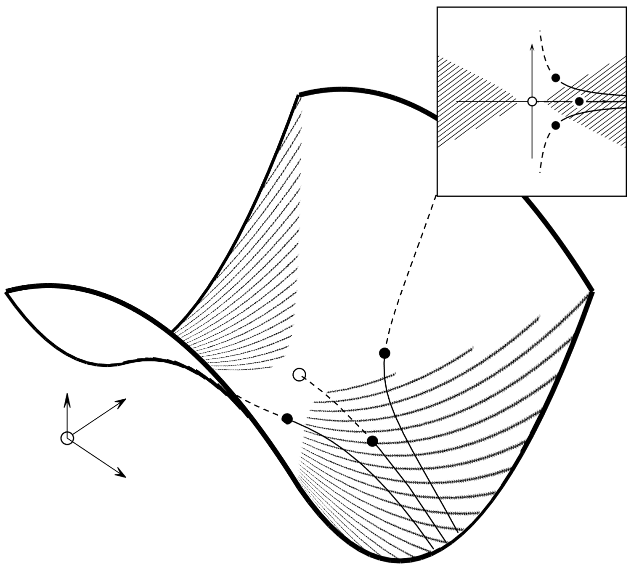

Figure 6.7: When z lies in the right half-plane (here any of the three black circles), the path of integration from $k = z$ to $k = \infty$ is deformed to the contours shown in the figure. In this case, the steepest descent procedure indicates that the dominant contribution comes solely from the endpoint $k = z$. The inset corresponds to the constant contours of $\mathrm{Im}(f)$ in the complex k-plane. The surface plot corresponds to $(\mathrm{Re}\, k, \mathrm{Im}\, k\, \mathrm{Re}\, f(k))$. Valleys are shown hatched.

The dominant contribution to the integral can be derived by expanding the integrand about its endpoints using integration by parts. After one integration of (6.36), we have

$$\phi(z) = \frac{\epsilon^2 \mathrm{e}^{-\frac{z^2}{2\epsilon^2}}}{z} - \int_z^\infty \frac{\epsilon^2 \mathrm{e}^{f(k)/\epsilon^2}}{f'(k)}\, \mathrm{d}k.$$

The full asymptotic expansion can be established by induction, yielding the asymptotic expansion of

$$\phi(z) \sim \mathrm{e}^{-\frac{z^2}{2\epsilon^2}} \sum_{n=0}^{\infty} \epsilon^{2n+2} \left[\frac{(-1)^n (2n)!}{2^n n! z^{2n+1}} \right] \equiv \phi_{\text{base}}(z). \tag{6.37}$$

Now consider the analytic continuation of $\phi(z)$ as z follows an anticlockwise trajectory about the origin in the k-plane. We realise that with

z in the left half-plane, and where its argument satisfies $\pi/2 < \theta < 3\pi/2$, now the steepest descent path connected to z fundamentally changes. This is shown in fig. 6.8. With the endpoint $k = z$ positioned in the left half-plane, in order to proceed to $k = +\infty$, the contour of steepest descent must first be deformed to the valley at $k = -\infty$, along $\text{Im}[f(k)] = \text{Im}[f(z)]$. From there, the integration contour must then traverse across the saddle point, $k = 0$, and along $\text{Im}[f(k)] = \text{Im}[f(0)]$, before arriving in the final valley of $k = +\infty$.

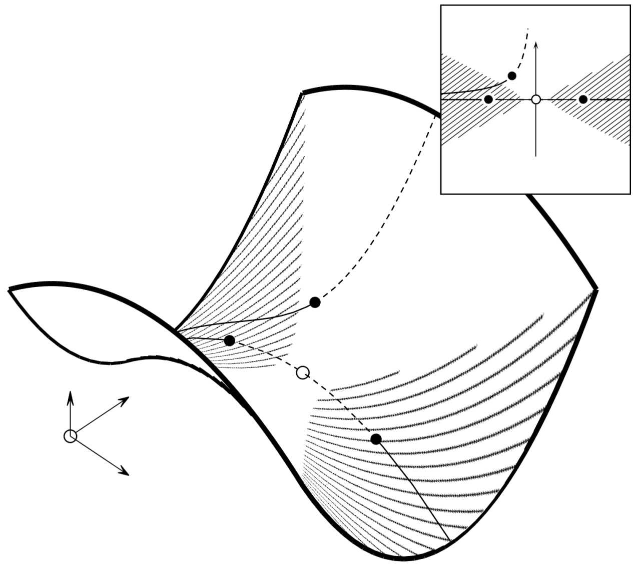

Figure 6.8: With z now in the left half-plane, the deformation of the initial contour of integration must first tend to the valley where $k = -\infty$, and the secondly through the saddle point in order to reach $k = +\infty$.

Since the second path of steepest descent lies exactly along the real k-axis, in this case, we must include the contribution

$$\int_{-\infty}^{\infty} \mathrm{e}^{f(k)/\epsilon^2}\,\mathrm{d}k = \int_{-\infty}^{\infty} \mathrm{e}^{f(k)/\epsilon^2}\,\mathrm{d}k = \sqrt{2\pi\epsilon}, \tag{6.38}$$

with the series (6.37).

The above demonstrates the occurrence of the *Stokes phenomenon*: here, the dominant base asymptotic expansion, $\phi_{\text{base}}(z)$, has switched on a subdominant exponentially-small contribution, $\sqrt{2\pi\epsilon}$, as z crosses $\arg(z) = \pi/2$. As with our introduction of the terminology, this is the *Stokes line* in section 6.1.1. If we continue the circular trajectory, a similar event occurs once z is analytically continued past the point $\theta = 3\pi/2$. In this case, we return to the first case studied in fig. 6.7 where the path of steepest descent proceeds directly from $k = z$ to $k = +\infty$. Hence the contribution (6.38) is switched off across the Stokes line with $\theta = 3\pi/2$.

In addition, we remark the conditions for Stokes line to occur in the steepest descent interpretation: Stokes lines correspond to those points, $z \in \mathbb{C}$, where (i) there is an equal phase contour connecting $k = z$ with the saddle point, $k = 0$, and (ii) the equal-phase contour is one of steepest descent. Thus, the corresponding conditions are

$$\text{Im}[f(z)] = \text{Im}[f(0)] = 0 \quad \text{and} \quad \text{Re}[f(z)] \geq \text{Re}[f(0)] = 0.$$

With $f(z) = -z^2/2$, this confirms that $\theta = \pi/2,\, 3\pi/2$ are Stokes lines.

If we include the first term of each respective contribution, we have the following asymptotic representation from (6.37) and (6.38):

$$\phi(z) \sim \mathrm{e}^{-z^2/(2\epsilon^2)} \times \begin{cases} \epsilon/z & \text{for } 0 < \theta < \pi/2 \\ \epsilon/z + \epsilon\sqrt{2\pi}\mathrm{e}^{z^2/(2\epsilon^2)} & \text{for } \pi/2 < \theta < 3\pi/2 \\ \epsilon/z & \text{for } 3\pi/2 < \theta < 2\pi \end{cases},$$

which should make it clear that the switched-on contribution is exponentially smaller than the first for z at the point it is switched-on or off.

Furthermore, notice that while the two Stokes lines $\theta = \pi/2, 3\pi/2$ correspond to the switching-on and switching-off of the exponentially-small term, $\sqrt{2\pi}\epsilon$, in fact the two contributions, ϕ_{base} and $\epsilon\sqrt{2\pi}$, will exchange dominance during the above analytic continuation procedure. This occurs at the *Anti-Stokes* lines, where $\mathrm{Re}[f(z)] = \mathrm{Re}[f(0)]$.

This is illustrated in fig. 6.9. Take note that this is now drawn in the z-plane. Notice in addition how we have specified the order of the two distinct contributions in each of the relevant sectors in order to emphasise the dominant contribution.

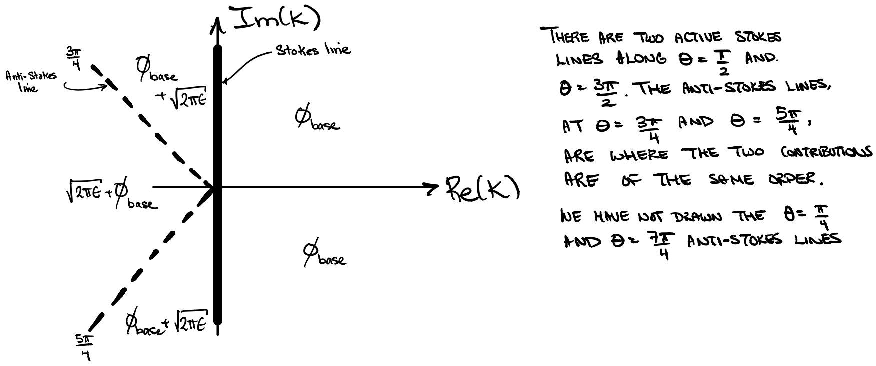

Figure 6.9: Layout of Stokes and Anti-Stokes lines. Typo in the image; should be the complex z-plane rather than the k-plane.

As expected, our analysis of the integral representation of ϕ using the method of steepest descents readily yielded the occurrence of the Stokes phenomenon, along with locations of the Stokes lines. The more mysterious process involves the analysis of the differential equation. This we do next.

6.3 Asymptotic expansion and WKBJ ansatz

Our strategy, like for the exponential integral, is to demonstrate the divergence of the naive asymptotic expansion for ϕ as it occurs in the singular differential equation (6.34), and moreover, to derive the Stokes phenomenon that necessarily occurs along the imaginary z-axis. Examining the form of the integral representation (6.35), and noting that the largest contribution as $\epsilon \to 0$ occurs at the left boundary of the integral, we intuit that the leading-order approximation of ϕ includes an exponential scaling proportional to $\epsilon\mathrm{e}^{-z^2/(2\epsilon^2)}$ from (6.37). Hence this suggests the use of a WKBJ ansatz [Bender and Orszag, 1999]. We substitute

$$\phi = \epsilon^2 A(z) \mathrm{e}^{-u(z)/\epsilon^2}, \tag{6.39}$$

into (6.34a), which yields:

$$\epsilon^2 \left[\frac{(-u')^2}{\epsilon^4} A - \frac{2u'}{\epsilon^2} A' - \frac{u''}{\epsilon^2} A + A'' \right] \mathrm{e}^{-u/\epsilon^2} + z \left[-\frac{u'}{\epsilon^2} A + A' \right] \mathrm{e}^{-u/\epsilon^2} = 0. \tag{6.40}$$

Considering the terms at leading order, $\mathcal{O}(1/\epsilon^2)$, we have $(u')^2 A - zu'A = 0$. Hence discounting[6] the two trivial solutions of $A = 0$ and $u =$ constant, we conclude that

[6] Since these would invalidate our *a priori* assumption of the dominant scaling.

$$u(z) = \frac{z^2}{2}, \tag{6.41}$$

where a constant of integration in u has been absorbed into the prefactor A. Substitution into (6.40) then yields

$$\delta A'' - zA' - A = 0, \tag{6.42}$$

where henceforth we set $\epsilon^2 = \delta$ for a more standardised asymptotic progression. Now that the leading exponential has been scaled from ϕ, the differential equation (6.42) presents a more typical asymptotic expansion in powers of δ.

We thus set

$$A(z) = \sum_{n=0}^{\infty} \delta^n A_n(z), \tag{6.43}$$

we obtain the recurrence relations

$$A_0(z) = \frac{a_0}{z}, \qquad A_n(z) = \frac{A'_{n-1}(z)}{z}, \quad n \geq 1, \tag{6.44}$$

where a_0 is constant. It can be verified, by the matching of inner-and-outer asymptotic solutions near $z = 0$, that the boundary condition (6.34b) requires $a_0 = 1$.

At this point, we recognise a similar trend to what has been observed in the exponential integral. Namely, the leading-order solution, $A_0 = 1/z$ is singular at $z = 0$. Each subsequent order involves differentiation of the previous order and a further division by z. Thus, $A_1 = -1/z^2$, $A_2 = 2/z^5$, $A_3 = \ldots$ and A_n is expected to diverge in the form of a factorial over power. Investigating the pattern, we find that

$$A_n(z) = \frac{(-1)^n (2n-1)!!}{z^{2n+1}}. \tag{6.45}$$

In fact, we may use the identity $(2n-1)!! = (2^n/\sqrt{\pi})\Gamma(n + 1/2)$ [cf. eqn (13.28) of Arfken et al. [2013]] to write A_n in the more typical factorial-over-power form of

$$A_n(z) = \frac{1}{\sqrt{-2\pi}} \frac{\Gamma(n + 1/2)}{(-z^2/2)^{n+1/2}}. \tag{6.46}$$

In order to be consistent with the notation we use throughout this book, we shall write the above in our typical factorial-over-power form given in the following. [7]

[7] It is important to note that in general, we shall seek to characterise asymptotic divergence as $n \to \infty$ using such factorial-over-power ansatzes. It is specifically for this problem that the general form of (6.47) satisfies the exact values of A_n for all n. This will not be the case in general.

Late terms of the complementary error function

The (late) terms of the asymptotic expansion for the complementary error function, governed by the ODE (6.42), is given by

$$A_n(z) = \frac{\Lambda\Gamma(n+\gamma)}{[\chi(z)]^{n+\gamma}}, \tag{6.47}$$

where $\Lambda = \sqrt{-1/(2\pi)}$, $\gamma = 1/2$, and $\chi(z) = -z^2/2$.[a]

[a]Note that we can select the square root branch and write $\Lambda = \mathrm{i}/\sqrt{2\pi}$, as long as we are also careful with the eventual factor of $(-1)^{1/2}$ in the expression for $\chi^{1/2}$.

6.3.1 Optimal truncation and Stokes line smoothing

At this point, we have successfully derived all the terms of the traditional asymptotic expansion of $\phi(z)$ and $A(z)$ and this has essentially re-derived (6.37) using only the differential equation. As a test, we may examine the error in the asymptotic approximation as a function of the truncation point. We calculate the Nth remainder given by

$$\mathcal{R}_N(z) = \phi(z) - \sum_{n=0}^{N-1} \epsilon^{2n} A_n(z) \mathrm{e}^{-z^2/(2\epsilon^2)}. \tag{6.48}$$

The numerically "exact" value of $\phi(z)$ is taken to be the value returned by a program such as Mathematica[8], and the remainder or error is shown in fig. 6.10. As illustrated, the graph demonstrates the divergence of the asymptotic series, and also the prototypical initial convergence towards an optimal truncation point. In this case, the optimal truncation point occurs on the twelfth term of the series.

[8]Though these programs will often use, themselves, asymptotic series for calculation. It is always wise to be cautious with the use of black-box special function implementations.

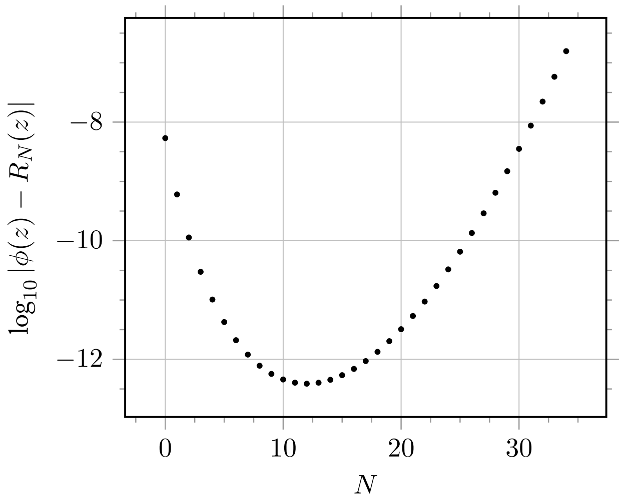

Figure 6.10: The error between exact $\phi(z)$ and the N-term truncated remainder, $\mathcal{R}_N(z)$. Here $\epsilon^2 = 0.2$ and $z = 1$. Observe the series divergence trend showing the optimal truncation point. The optimal truncation point is on the $N = 13$ term.

In general[9], the minimum of the curve shown in the figure is where the difference in magnitude of successive terms in the series is smallest, i.e.

$$\left|\frac{\epsilon^{2N+1} A_{N+1}}{\epsilon^{2N} A_N}\right| = \left|\frac{\delta^{N+1} A_{N+1}}{\delta^N A_N}\right| \sim 1. \tag{6.49}$$

[9]Throughout this work, we shall operate on a number of rules-of-thumbs that are widely applicable; the rule-of-thumb in this case that optimal truncation occurs where the smallest term of the series is retained can be rigorously demonstrated for certain classes of functions [cf. Jones, Berry, Howls, etc.]

In this case, we select the nearest integer, rounded up, given by

$$N = \frac{|\chi|}{\delta} + \rho = \frac{z^2}{2\delta} + \rho, \tag{6.50}$$

where $0 \le \rho < 1$. For example, substitution of $z = 1$ and $\delta = \epsilon^2 = 0.2^2$ verifies that $N \approx 12.5$, as observed in fig. 6.10.

We now seek to determine the optimally truncated remainder. We set

$$A(z) = \sum_{n=0}^{N-1} \delta^n A_n(z) + R_N(z),$$

into (6.42), noting that this is a slightly different remainder than in (6.48). This yields

$$\mathcal{L}R_N = -\epsilon^{2N} A''_{N-1} = -\epsilon^{2N}(zA'_N + A_N) \sim -\epsilon^{2N} z A'_N,$$

where $\mathcal{L}R_N \equiv \epsilon^2 R''_N - zR'_N - R_N$. In the last similarity, we have used the divergence property of A_N as $N \to \infty$. Substituting (6.47) into the above, we obtain

$$\mathcal{L}R_n \sim -\delta^N z \frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\Lambda\Gamma(N+\gamma)}{\chi^{N+\gamma}}\right\} = -\frac{\delta^n (\chi')^2 \Lambda\Gamma(N+\gamma+1)}{\chi^{N+\gamma+1}}. \tag{6.51}$$

In the last equality, we have used $\chi' = -z$. During these derivations, it is often easiest to work with χ instead of its explicit form as a function of z.

Our procedure is similar now to the one performed for the Exponential Integral in section 6.1.2. That is, we shall localise the above differential equation near the suspected Stokes lines, where χ is real and positive, and examine the jump in the remainder as the Stokes line is crossed.

It is convenient to scale out the homogeneous solution, expressing

$$R_N(z) = S(z)\Lambda \mathrm{e}^{-\chi/\delta}. \tag{6.52}$$

Doing so, we obtain[10]

$$\mathcal{L}R_N = \Lambda \mathrm{e}^{-\chi/\delta}\Big[\delta S'' - \chi' S' - \chi'' S\Big]. \tag{6.53}$$

[10] There seems to be some typos in the below, which may explain the typo in the final result—needs to be fixed Nov 2024

Now our task is to set $\chi = r\mathrm{e}^{\mathrm{i}\vartheta}$ and examine the rate of change of $S(z)$ as z crosses the critical line. Like the exponential integral, the critical scaling for the Stokes line will be with $\vartheta = \mathcal{O}(\sqrt{\delta})$. The change in S is significant in the boundary layer, and it can be verified *a posteriori* that $S' \gg \delta S'', S$. For simplicity of the presentation, we shall thus assume this to be the case at this stage.

Combining (6.51) with (6.53) we then have

$$\frac{\mathrm{d}S}{\mathrm{d}\chi} \sim \mathrm{e}^{\chi/\delta}\delta^N \frac{\Gamma(N+\gamma+1)}{\chi^{N+\gamma+1}}.$$

We may now substitute $\chi = r\mathrm{e}^{\mathrm{i}\vartheta}$ and thus $\frac{\mathrm{d}}{\mathrm{d}\chi} = -(\mathrm{i}/\chi)\frac{\mathrm{d}}{\mathrm{d}\vartheta}$. Then

$$\frac{\mathrm{d}S}{\mathrm{d}\vartheta} \sim \frac{\mathrm{i}\chi \mathrm{e}^{\chi/\delta}\delta^N \Gamma(N+\gamma+1)}{\chi^{N+\gamma+1}}. \tag{6.54}$$

Previously, using Stirling's approximation for the Gamma function, we derived the approximation (6.23). We need only substitute $\epsilon \mapsto \delta$ and $\gamma \mapsto \gamma + 1$ in this expression.

Consequently we have

$$\frac{\mathrm{d}S}{\mathrm{d}\vartheta} \sim \frac{\sqrt{2\pi r}\mathrm{i}}{\delta^{\gamma+1/2}} \exp\left[\frac{r\mathrm{e}^{\mathrm{i}\vartheta}}{\delta} - \frac{r}{\delta} - (N+\gamma)\mathrm{i}\vartheta\right]. \tag{6.55}$$

We let $\chi = r\mathrm{e}^{\mathrm{i}\vartheta}$ and change derivatives via $\frac{\mathrm{d}}{\mathrm{d}z} = \chi'/(\mathrm{i}\chi)\frac{\mathrm{d}}{\mathrm{d}\vartheta}$. Then we have The reader should compare this with (6.27).

At this point, we can see that the rate of change of S is typically exponentially small except when χ is real and positive. Indeed, if we expand about $\vartheta = 0$, we find

$$\frac{\mathrm{d}S}{\mathrm{d}\vartheta} \sim \frac{\sqrt{2\pi r}\mathrm{i}}{\delta^{\gamma+1/2}} \mathrm{e}^{r\vartheta^2/(2\delta)}, \tag{6.56}$$

which reveals the expected scaling for ϑ. Setting $\vartheta = \sqrt{\delta}\bar{\vartheta}$ gives the jump across the Stokes line as

$$[S] \sim \frac{\sqrt{2\pi r}\mathrm{i}}{\delta^{\gamma}} \int_{-\infty}^{\infty} \mathrm{e}^{-r\bar{\vartheta}^2/2} \,\mathrm{d}\bar{\vartheta} = \frac{2\pi\mathrm{i}}{\delta^{\gamma}}. \tag{6.57}$$

Therefore across the Stokes line, the expected remainder is

$$R_N \sim \frac{2\pi\mathrm{i}}{\delta^{\gamma}} \Lambda \mathrm{e}^{-\chi/\delta} = -\frac{\sqrt{2\pi}}{\epsilon} \mathrm{e}^{z^2/(2\epsilon^2)}, \tag{6.58}$$

using $\gamma = 1/2$, $\delta = \epsilon^2$, $\Lambda = \mathrm{i}/\sqrt{2\pi}$, $\chi = -z^2/2$. Remembering that the remainder is multiplied by $\epsilon^2 \mathrm{e}^{z^2/2\epsilon^2}$ to give the solution in ϕ, we see that it now matches the expected exponential of $\phi_{\exp} \sim \sqrt{2\pi}\epsilon$.

Note: there is a sign error in the above; it may have to do with our interpretation of $\Lambda = \sqrt{-1/2\pi}$.

6.4 EXERCISES

1. Derive the asymptotic simplification leading to (6.23).
2. With $R_N = \mathcal{S}\mathrm{e}^{-\chi/\epsilon}$ and from (6.24) repeated below:

   $$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z} \sim \left[\sqrt{\frac{2\pi}{|\chi|}} \frac{1}{\epsilon^{\gamma+1/2}}\right] \frac{\mathrm{e}^{\chi/\epsilon}\mathrm{e}^{-|\chi|/\epsilon}}{(\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}\,\chi})^{N+\gamma}}, \tag{6.59}$$

   demonstrate that the above ODE is associated with a Stokes switching of the form

   $$R_N \sim \frac{2\pi\mathrm{i}}{\epsilon^{\gamma}} \mathrm{e}^{-\chi/\epsilon}. \tag{6.60}$$

   This requires performing the calculation shown in section 6.1.3.
3. Consider the ODE for the complementary error function problem given in

   By substituting the factorial-over-power ansatz (6.47) directly into the recurrence relation for A_n in (6.44), derive the form of χ.

   Show furthermore that $\gamma = 1/2$ by comparing the exact value of $A_1 = -1/z^3$ with the above prescribed form of A_n when $n = 1$.