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4

The asymptotic analysis of a differential equation is often challenging because of the competition between local and global features: the differential equation provides a local relationship between the values of a function and its derivatives at a point. Thus, a local analysis (like the expansion of the solution into a series) typically yields unknown constants of integration—these must then be related to boundary- or initial conditions through a global study of the solution. As we saw in the previous chapter on the method of matched asymptotic expansions, in many singular problems, the asymptotic expansion becomes disordered in different parts of the domain, rendering this task difficult.

In some cases, it is possible to formulate the problem as an integral. This formulation may be natural (*e.g.* a physical law posed as an integral) or through a further transformation of the differential equation. Integral formulations are often simpler because the global information of the problem is explicitly specified within the integrand, and the initial- or boundary-conditions embedded, with no need to solve for additional constants. The task, then, is to perform an asymptotic analysis of the integrals.

As we shall see in later chapters, many exponential asymptotic investigations are done by selecting a problem with an integral formulation, developing the asymptotics of the integral, before returning to the differential equation and developing the parallel theory, there, pretending to be ignorant of the previous findings.

4.1 INTEGRATION BY PARTS

For certain problems, simple integration-by-parts allows a derivation of the asymptotic expansion. These problems are characterised by those where the dominant contributions to the integrals are primarily from the endpoints. The following is a standard example of this phenomenon.

Example 4.1 (Integration by parts for Ei(x))
We seek an asymptotic expansion of the exponential integral,

$$\mathrm{Ei}(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \tag{4.1}$$

in the limit $x \to \infty$. With $\mathrm{d}v = e^{-t}\,\mathrm{d}t$ and $u = -1/t$, integration by parts yields

$$\begin{aligned}\mathrm{Ei}(x) &= \left[-\frac{\mathrm{e}^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{-x}}{x} + \left[\frac{\mathrm{e}^{-t}}{t^2}\right]_x^\infty + 2\int_x^\infty \frac{\mathrm{e}^{-t}}{t^3}\,\mathrm{d}t.\end{aligned}$$

This process can be continued in order to develop the divergent series

$$\mathrm{Ei}(x) \sim \mathrm{e}^{-x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{x^n}. \tag{4.2}$$

It can be useful to visualise the integrand in the asymptotic limit. Let $x = X/\epsilon$ and consider fixed X and $\epsilon \to 0$. Then with $t = x\epsilon$, the integral transforms to

$$\mathrm{Ei}(X/\epsilon) = \epsilon \int_X^{\infty} \frac{\mathrm{e}^{-s/\epsilon}}{s} \, \mathrm{d}s. \tag{4.3}$$

The integrand is visualised in fig. 4.1 for three different values of ϵ. As $\epsilon \to 0$, the integrand becomes exponentially suppressed everywhere away from $s = 0$, but with the largest contribution to the integral near the endpoint of integration, $s = X$.

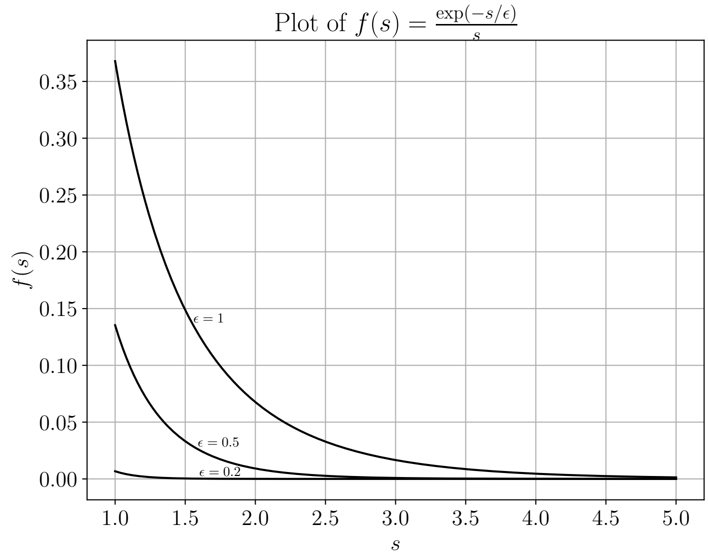

Figure 4.1: Illustration of the integrand $f(s) = \mathrm{e}^{-s/\epsilon} s$ for different values of ϵ. In these graphs, the endpoint is chosen to be $s = X = 1$.

The following is an example where the integral, in its original form, must be re-written so that integration-by-parts can be applied.

Example 4.2 (Integration by parts with $\infty - \infty$)
We consider an asymptotic expansion of the integral

$$I(x) = \int_0^x t^{-1/2} \mathrm{e}^{-t} \, \mathrm{d}t,$$

in the limit $x \to \infty$.

Naively integrating by parts yields

$$I(x) = \left[-t^{-1/2} \mathrm{e}^{-t} \right]_0^x - \frac{1}{2} \int_0^x t^{-3/2} \mathrm{e}^{-t} \, \mathrm{d}t,$$

and thus the indeterminate value of $\infty - \infty$. However, we may remove the leading-order contribution by first re-writing,

$$I(x) = \int_0^\infty t^{-1/2} \mathrm{e}^{-t} \, \mathrm{d}t - \int_x^\infty t^{-1/2} \mathrm{e}^{-t} \, \mathrm{d}t. \tag{4.4}$$

The first integral has value $\Gamma(1/2) = \sqrt{\pi}$. The second integral can now be integrated by parts, with

$$I(x) = \sqrt{\pi} + \int_x^\infty t^{-1/2} \frac{\mathrm{d}\mathrm{e}^{-t}}{\mathrm{d}t} \, \mathrm{d}t = \sqrt{\pi} + \frac{\mathrm{e}^{-x}}{\sqrt{x}} + \frac{1}{2} \int_x^\infty \mathrm{e}^{-3/2} \mathrm{e}^{-t} \, \mathrm{d}t,$$

and the process can be continued to higher order. Integration by parts typically works if the dominant contribution to the integral comes from the endpoints. The trick of writing the integral in the form (4.4) works for this problem; the leading-order estimate is the complete integral over $t \in [0, \infty)$—naive integration by parts yields an alternating series of infinities because of the $t^{-1/2}$ factor.

4.2 LAPLACE'S METHOD

Laplace's method is a general technique for obtaining the asymptotic expansion of integrals of the form,

$$I(x) = \int_a^b f(t) \mathrm{e}^{x\phi(t)} \, \mathrm{d}t, \tag{4.5}$$

in the limit $x \to \infty$. Such integrals are quite common in applications related to wave propagation. For now, we consider real-valued functions, f and ϕ. The allowable f and ϕ are those which make the integral exist for the given x.

Example 4.3 (Laplace integral with boundary contribution) Consider the asymptotic expansion of

$$I(x) = \int_0^1 \frac{\mathrm{e}^{-xt}}{1+t} \, \mathrm{d}t,$$

as $x \to \infty$.

Observe that as $x \to \infty$, the largest contribution to the integral comes from $t = 0$, since the factor e^{-xt} causes a very fast exponential decay away from the origin. This is visualised in fig. 4.2.

We thus split the range of integration as

$$I(x) = \int_0^\epsilon \frac{\mathrm{e}^{-xt}}{1+t} \, \mathrm{d}t + \int_\epsilon^1 \frac{\mathrm{e}^{-xt}}{1+t} \, \mathrm{d}t.$$

where $1/x \ll \epsilon \ll 1$. The second integral is bounded by its contribution at $t = \epsilon$, and

$$\left| \int_\epsilon^1 \frac{\mathrm{e}^{-xt}}{1+t} \, \mathrm{d}t \right| \le \mathrm{e}^{-x\epsilon} \int_0^1 \frac{1}{1+t} \, \mathrm{d}t = \mathrm{e}^{-x\epsilon} \log 2.$$

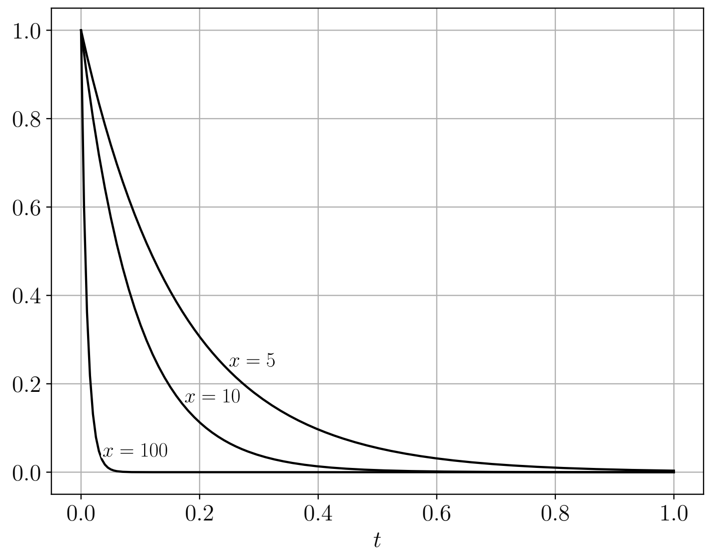

Figure 4.2: A plot of the integrand $e^{-xt}/(1+t)$ for increasing values of x. As $x \to \infty$, the dominant contribution to the integral comes from the endpoint, $t = 0$.

so is $\mathcal{O}(e^{-\epsilon x})$ and is neglected from exponential smallness.

Returning to the first integral, we remark the size of the factor e^{-xt} is difficult to interpret since both $x \to \infty$ and $t = \epsilon \to 0$. Therefore, let us set $s = xt$. Then

$$I(x) \sim \frac{1}{x} \int_0^{x\epsilon} \frac{e^{-s}}{1 + s/x} \, ds.$$

Since the largest value of s, $x\epsilon$, is $\ll x$, we may now expand the denominator via a standard geometric series:

$$I(x) \sim \frac{1}{x} \int_0^{x\epsilon} e^{-s} \sum_{n=0}^{\infty} \frac{(-s)^n}{x^n} \, ds.$$

The interior series expansion is uniform on the domain $0 < s < \epsilon x$, so we may interchange integration and summation to give

$$I(x) \sim \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \int_0^{x\epsilon} (-s)^n e^{-s} \, ds.$$

A key trick now occurs in Laplace's method. We can essentially extend the upper limit of integration from $x\epsilon$ to infinity. This is since

$$\int_0^{x\epsilon} (-s)^n e^{-s} \, ds = \left(\int_0^{\infty} - \int_{x\epsilon}^{\infty} \right) (-s)^n e^{-s} \, ds.$$

However, the rightmost integral is exponentially small, and its magnitude is of order $(x\epsilon)^n e^{-x\epsilon}$. Therefore the approximation of

$$I(x) \sim \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \int_0^{\infty} (-s)^n e^{-s} \, ds,$$

introduces only exponentially small errors. Finally, by definition of the gamma function:

$$\int_0^\infty s^n e^{-s}\, ds = \Gamma(n+1) = n!,$$

we finally have

$$I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \dots$$

The previous example 4.3 was an example where the dominant contribution of the Laplace-type integral (4.5) occurs at the endpoint of integration. Another interesting case occurs when the dominant contribution occurs within the interior of the domain of integration. This is discussed below, but we shall find it easier to discuss the method in the context of general integrals.

Consider again the Laplace-type integral,

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\, dt, \tag{4.6}$$

in the limit $x \to \infty$. It is assumed that x is real and positive, that g is a real continuous function, and ϕ, ϕ', and ϕ'' are real and continuous on $t \in [a, b]$.

The essence of Laplace's idea is that as $x \to \infty$, the dominant contribution to the integral comes from the neighbourhood of the point in the integration domain where ϕ is maximal. Hence there are primarily two cases to consider: (i) ϕ is maximal at its endpoints, $t = a$ or $t = b$; or (ii) ϕ is maximal at an interior point, $t = c$, with $c \in (a, b)$. In both of these cases, the procedure is as follows.

First, the range of integration is limited to a neighbourhood of the maximum, say $t = t^*$, of ϕ; contributions to I outside of this neighbourhood are argued to be subdominant (typically exponentially subdominant). Second, the factor f and ϕ are expanded into Taylor series about $t = t^*$. Third, one or both integration limits are extended to infinity, at the cost of only introducing exponentially small errors. The resultant integral then does not depend on the precise specification of the neighbourhood, and produces the asymptotic expansion of I.

We examine the case of a maximum within the interior (case ii above), leaving the general argument of the maximum at the endpoint (case i) for the exercises. Let us thus consider (4.6) split into a local and nonlocal part:

$$I(x) = \left(\int_a^{c-\epsilon} + \int_{c-\epsilon}^{c+\epsilon} + \int_{c+\epsilon}^{b} \right) f(t) e^{x\phi(t)}\, dt,$$

where $c \in (a, b)$ is assumed to be the maximum value of ϕ over the interval $[a, b]$. The exact size of ϵ is left undetermined for now, but it will certainly be such that $\epsilon \to 0$ as $x \to \infty$. Then, by assumption of c and continuity of the integrand, the first integral is $\mathcal{O}(e^{x\phi(c-\epsilon)})$, and the third is $\mathcal{O}(e^{x\phi(c+\epsilon)})$. Note that

$$\phi(c+\epsilon) = \phi(c) + \frac{\epsilon^2}{2}\phi''(c) + \dots,$$

since ϕ is a maximum at $t = c$. Thus the exponential factor is

$$e^{[x\phi(c+\epsilon)]} = e^{x\phi(c)} e^{x\phi''(c)\epsilon^2/2+\dots}$$

and is exponentially smaller than $e^{x\phi(c)}$ if

$$x\epsilon^2 \ll 1. \tag{4.7}$$

A similar argument applies for the exponential with factor $\phi(c - \epsilon)$.

Examining only the second integral, we then expand its arguments about $t = c$:

$$I \sim \int_{c-\epsilon}^{c+\epsilon} \Big[f(c) + f'(c)(t-c) + \dots\Big] \times e^{x[\phi(c)+\phi''(c)/2(t-c)^2+\phi'''(c)/3!(t-c)^3+\dots]}\, dt. \tag{4.8}$$

This can be written as

$$I \sim \int_{c-\epsilon}^{c+\epsilon} \Big[f(c) + f'(c)(t-c) + \dots\Big] e^{x\phi(c)} e^{x\phi''(c)(t-c)^2/2} \times e^{x[\phi'''(c)/3!(t-c)^3+\dots]}\, dt. \tag{4.9}$$

It is convenient to perform a coordinate transformation in order to remove x from the exponential. Since the second exponential on the first line is of Gaussian form, this suggests setting $s^2 = x(t-c)^2$ and hence $\sqrt{x}(t - c) = s$. Then

$$I \sim \frac{e^{x\phi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\epsilon}^{\sqrt{x}\epsilon} \Big[f(c) + f'(c)\frac{s}{\sqrt{x}} + \dots\Big] e^{s^2\phi''(c)/2} \times e^{\left[\frac{\phi'''(c)}{6\sqrt{x}} s^3+\dots\right]}\, ds. \tag{4.10}$$

Within the integration domain, the last exponential factor can be expanded into a Taylor series as long as

$$\left|\frac{s^3}{x^{1/2}}\right| \le \left|\frac{(x^{1/2}\epsilon)^3}{x^{1/2}}\right| = |x|\epsilon^3 \ll 1.$$

Thus in combination with the condition (4.7), in our selection of the neighbourhood of $t = c$, we select the size:

$$\frac{1}{x^{1/2}} \ll \epsilon \ll \frac{1}{x^{1/3}}.$$

Returning to the integral (4.10), we consider only the leading term for simplicity. We have

$$I \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\epsilon}^{\sqrt{x}\epsilon} e^{s^2\phi''(c)/2}\, ds.$$

In this expression, we may finally let the upper and lower limits of integration tend to infinity, only introducing exponentially small errors. Thus

$$I(x) \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}} \int_{-\infty}^{\infty} e^{s^2\phi''(c)/2}\, ds = \frac{\sqrt{2\pi} f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}. \tag{4.11}$$

Example 4.4 (Laplace integral with interior contribution)
Consider the asymptotic expansion of

$$I(x) = \int_0^1 \frac{\mathrm{e}^{-xt}}{1+t}\,\mathrm{d}t$$

as $x \to \infty$.

Example 4.5 (Stirling's formula)
Consider the asymptotic expansion of

$$\Gamma(x+1) = \int_0^\infty t^x \mathrm{e}^{-t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t+x\log t}\,\mathrm{d}t,$$

as $x \to \infty$.

Hint: consider the transformation $t = x\tau$.

Under the transformation, we have

$$\Gamma(x+1) = x^{x+1}\int_0^\infty \mathrm{e}^{x\phi(\tau)}\,\mathrm{d}\tau, \qquad \phi(\tau) = -\tau + \log\tau.$$

There is a single maximum at $\tau = 1$ where $\phi(1) = -1$ and $\phi''(1) = -1$. Thus using the formula (4.11) yields

$$\Gamma(x+1) \sim x^{x+1}\mathrm{e}^{-x}\sqrt{\frac{2\pi}{x}} = \sqrt{2\pi}x^{x+1/2}\mathrm{e}^{-x}.$$

If $x = n = 0, 1, 2, \ldots$ is a non-negative integer, then $\Gamma(x+1) = n!$ and the above approximation yields the celebrated *Stirling's formula*:

$$n! \sim \sqrt{2\pi}n^{n+1/2}\mathrm{e}^{-n}. \tag{4.12}$$

There are many extensions and generalisations of Laplace's method, but the basic idea is generally the same: approximate the integral by considering local expansions about points in the integrand that provide the dominant contributions. Variations on this theme can include: (i) expansion about points where the integrand function, $f(t) = 0$; this can often be remedied via initial integration-by-parts and Taylor expansion of f; (ii) movable maxima problems of the form $\int f(t)\mathrm{e}^{\phi(x,t)}\,\mathrm{d}t$ where now the location of the maxima changes with the value of x.

4.3 EXERCISES

1. Watson's lemma applies to integrals of the form

$$I(x) = \int_0^b f(t)\mathrm{e}^{-xt}\,\mathrm{d}t, \qquad b > 0.$$

Suppose that f is continuous on $t \in [0, b]$ and has the asymptotic expansion

$$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}, \qquad \text{as } t \to 0^+,$$

where $\alpha > -1$ and $\beta > 0$ so that the integral converges at $t = 0$. If $b = \infty$, then we also restrict $f \ll e^{ct}$ as $t \to \infty$ for some positive constant c so the integral converges. Then Watson's lemma states

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}, \qquad x \to \infty.$$

By following the procedure as in example 4.3, prove Watson's lemma.

5

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In generalising the integral approximation methods from the previous chapter to complex-valued functions, a very powerful method of asymptotic approximation, called the *method of steepest descents*, can be developed. This time, we are interested in the asymptotics of integrals of the form

$$I(\lambda) = \int_C f(t)\mathrm{e}^{\lambda h(t)}\,\mathrm{d}t, \tag{5.1}$$

where C is some contour in the complex plane, f and h are complex-valued (typically holomorphic) functions of $t \in \mathbb{C}$, and $\lambda > 0$ is real. We are primarily interested in the limit of $\lambda \to \infty$. Note that the contour, C, can be between finite points in the plane, but can also be infinite or semi-infinite in extent.

Let us write the real and imaginary parts of h as

$$h(t) = \phi(t) + \mathrm{i}\psi(t).$$

If the contour C runs between points t_a to t_b, we have

$$|I(\lambda)| \leq \int_{t_a}^{t_b} |f|\mathrm{e}^{\lambda\phi(t)}\,\mathrm{d}t < L\max_L(|f|\mathrm{e}^{\lambda\phi}),$$

where the max notation corresponds to the maximum of the argument along the path of integration C, and L corresponds to the length of C. Following Laplace's method, the argument seems straightforward: consider the integral (5.1) in a neighbourhood of the maximal value of ϕ, say at $t = t_0$. Then from the previous chapter,

$$I(\lambda) \sim f(t_0)\sqrt{\frac{2\pi}{-\lambda h''(t_0)}}\mathrm{e}^{\lambda h(t_0)}.$$

However, in doing so, we find that the above estimate is far too large. The issue is that that there is, in general, a variable phase component, with factor $\mathrm{e}^{\mathrm{i}\lambda\psi(t)}$. In the limit $\lambda \to \infty$, this produces a highly oscillatory integrand, where there is significant positive and negative cancellation, which must be taken into account when integrating locally about $t = t_0$. The trick is to choose a path along which $\psi = \operatorname{Im} h(t)$ is constant—in this case, $\mathrm{e}^{\mathrm{i}\lambda\psi}$ is unchanging in t and the integrand behaves in the fashion we expect with its dominant contribution centred about $t = t_0$ and decaying in a well-behaved way away from this point.

We must consider the complex-valued landscape. We assume that away from points of non-analyticity, h is otherwise analytic and thus its real and imaginary parts satisfy the Cauchy-Riemann equations:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \tag{5.2}$$

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{5.3}$$

From definition of the complex derivative,

$$h'(z) = \phi_x + \mathrm{i}\psi_x = \phi_x - \mathrm{i}\psi_y. \tag{5.4}$$

Therefore at $z = z_0 = x_0 + \mathrm{i}y_0$, a relative maximum of ϕ, the two components are zero, $\phi_x = \phi_y = 0$. A similar relationship on the first derivative holds for ψ. Consequently such points correspond to critical points of h with

$$h(t_0) = 0. \tag{5.5}$$

Moreover, by the standard tenants of complex functions, ϕ and ψ are potential functions, satisfying Laplace's equation, $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$. By the maximum modulus theorem, ϕ and ψ cannot have extrema in the domain of analyticity of $h(z)$. Therefore, z_0 must be a saddle point of ϕ and ψ, and we will analogously say that h has the saddle point z_0.

It remains to consider the optimal path through z_0, for which we must deform C. As discussed earlier, we wish to choose paths for which $\psi = \operatorname{Im} h$ is constant in order to avoid the issue of rapid oscillations as $\lambda \to \infty$. By the Cauchy-Riemann quations,

$$\nabla\phi \cdot \nabla\psi = 0,$$

then it is the case that lines of constant ϕ are orthogonal to lines of constant ψ. By elementary calculus, $\nabla\phi$ is the direction of steepest descent/ascent. Together, with the above equality, this indicates that *the path for which ψ is constant is precisely the path of steepest descent/ascent for ϕ.*

Indeed this seems to be an optimal path indeed for which to apply Laplace's method.

The formula for the method of steepest descents is as follows. If the integration domain is t: (i) determine the endpoints and saddle points of h, i.e. points t_0 with $h'(t_0) = 0$. Other critical points, such as branch points or singularities, may need to be considered on a case-by-case basis. Next, (ii) determine the paths of steepest descent/ascent through such saddle points. These are given by curves, $t \in \mathbb{C}$, such that

$$\operatorname{Im} h(t) = \text{const.} \tag{5.6}$$

Typically, the crucial curve of interest is the one that travels through the saddle point in (i), i.e. $\operatorname{Im} h(t) = \operatorname{Im} h(t_0)$. A point on the path is considered on the point of a path of steepest descent if

$$\operatorname{Re} h(t) > \operatorname{Re} h'(t_0). \tag{5.7}$$

Next, (iii) evaluate the local contributions as in Laplace's method.

The above procedure is repeated for all the relevant points, which includes endpoints of integration and saddle points, and also branch points and singularities. The exact deformation procedure will always depend on the precise topology of the integrand function.

Example 5.1 (Steepest descent on inverse gamma)
Consider the asymptotic approximation of

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi \mathrm{i}} \int_C t^{-x} \mathrm{e}^t \, \mathrm{d}t,$$

in the limit $x \to \infty$.[1]

[1] Note we use x here as the large parameter, in connection to prior use of the gamma function.

Note that there is a branch cut of the logarithm corresponding to $t^{-x} = \mathrm{e}^{-x \log t}$, which we take to be along the negative real axis. Above, C is the so-called *Hankel contour*, which starts at $t = -\infty - \mathrm{i}a$, $a > 0$, encircles the branch cut, and returns to infinity along $t = -\infty + \mathrm{i}b$, $b > 0$.

This is a moveable maxima problem; writing

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{e}^{-x \log t + t} \, \mathrm{d}t.$$

If we naively consider the extrema of the complete exponential argument, we see that it contains a stationary point at $t = x$, which is moving as $x \to \infty$. To put the integral into a more standard form, we set $t = xs$ giving

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi \mathrm{i} x^{x-1}} \int_C \mathrm{e}^{xh(s)} \, \mathrm{d}s, \quad \text{where } h(s) = s - \log s.$$

Considering now the complex saddle points, we have $h'(s) = 1 - 1/s$, therefore there is a saddle point at $s = 1$. We expand:

$$h(s) \sim 1 + \frac{(s-1)^2}{2} - \frac{(s-1)^3}{3} + \dots \tag{5.8}$$

The usual change in integration variable allows the quadratic term to be placed into standard form. We set $x(s-1)^2 = u^2$ (taking the appropriate positive roots). Then

$$\begin{aligned} &\sim \frac{\mathrm{e}^x}{2\pi \mathrm{i} x^{x-1} x^{1/2}} \int_C \mathrm{e}^{\frac{u^2}{2} - \frac{u^3}{3\sqrt{x}} + \dots} \, \mathrm{d}u \\ &= \frac{\mathrm{e}^x}{2\pi \mathrm{i} x^{x-1} x^{1/2}} \int_C \mathrm{e}^{\frac{u^2}{2}} \left(1 + \mathcal{O}(u^3/\sqrt{x})\right) \mathrm{d}u. \end{aligned} \tag{5.9}$$

We can observe from the above that the steepest descent contour is parallel to the imaginary axis, and it is advantageous to set $u = \mathrm{i}v$ so as to turn the exponential into the standard Gaussian function. More systematically, however, we can set $s - 1 = s_x + \mathrm{i}s_y$. Then noting that

$$(s-1)^2 = (s_x^2 - s_y^2) + 2\mathrm{i} s_x s_y,$$

we see that the contour of steepest descent near $s = 1$ is given by

$$\operatorname{Im} h(s) \sim s_x s_y = \operatorname{Im} h(1) = 0,$$

and therefore the appropriate contour, which goes through $s_x + \mathrm{i}s_y = 0$, will locally follow the vertical line, with $s_x = 0$ corresponding to $\operatorname{Im} s = 1$.

The contours, as calculated from the $h(z)$ function are shown in fig. 5.1. Dashed lines correspond to lines of constant $\operatorname{Re} h = \phi$ and solid lines to steepest descent lines of constant $\operatorname{Im} h = \psi$. We observe the saddle-point structure near $z = 1$. The initial Hankel contour, which tends to $-\infty$ initially below the real axis, to $-\infty$ above the real axis, must thus be deformed into the steepest descent contour (solid) that passes through $z = 1$. Thus, we see that the new contour begins in a valley (shown grey), increases to the maximum $z = 1$, and decreases again to another valley.

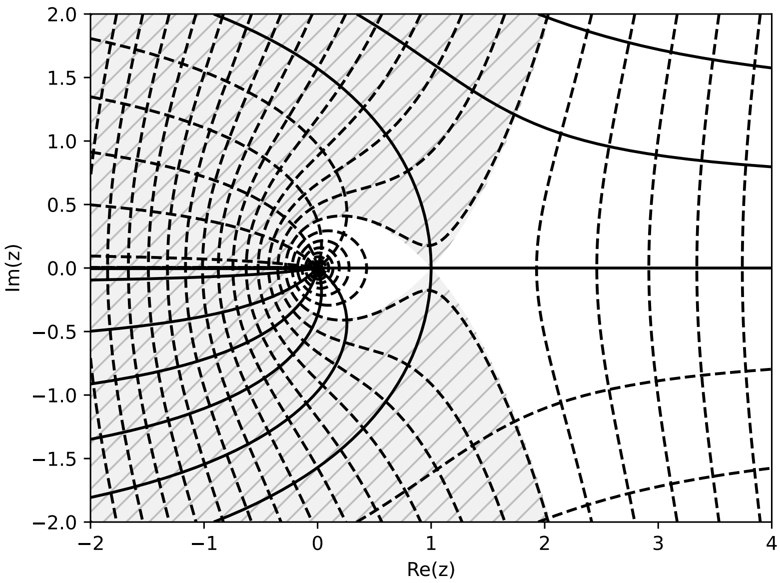

Figure 5.1: Contours of $\operatorname{Im} h = \psi$ (solid) and $\operatorname{Re} h = \phi$ (dashed). Grey regions indicate regions where $\operatorname{Re} h \leq \operatorname{Re} h(1)$, i.e. valleys with respect to the saddle point at $z = 1$.

Finally, returning to the integral (5.9) with $u = \mathrm{i}v$, we have

$$\frac{1}{\Gamma(x)} \sim \frac{\mathrm{e}^x}{2\pi x^{x-1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-v^2/2}\, \mathrm{d}v = \frac{\mathrm{e}^x}{\sqrt{2\pi} x^{x-1/2}}. \tag{5.10}$$

This matches the result for Stirling's formula established in example 4.5.