3

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At the beginning of chapter 2 and via (2.1), we introduced a complexity that often occurs in the study of singular boundary-value problems: the existence of *boundary layers*. Such boundary layers are characterised by regions where the solution derivatives are asymptotically large, and hence terms such as $\epsilon y''$ are no longer negligible compared to *e.g.* y. For such problems, the solution can be approximated by asymptotic expansions valid in different regions. We often refer to solutions valid within an 'outer' region, where the function and its derivatives are bounded as $\epsilon \to 0$. This is in contrast to an 'inner' region, valid within the boundary layers. Once the solutions are determined in their respective regions, they are then matched together in order to determine unknown constants of integration. It is easiest to explain the methodology via particular examples.

3.1 AN EXAMPLE WITH A BOUNDARY LAYER

Consider again (2.1):

$$\begin{gathered} \epsilon y'' + 2y' + 2y = 0, \qquad 0 < x < 1 \\ y(0) = 0 \quad \text{and} \quad y(1) = 1. \end{gathered} \tag{3.1}$$

Outer solution. We expand the solution as

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \dots \tag{3.2}$$

This yields at leading order,

$$2y_0' + 2y_0 = 0. \tag{3.3}$$

We solve the leading-order equation, and based on the intuition of fig. 2.1, we impose the boundary condition at $x = 1$ and hence $y_0(1) = 1$. This yields the leading-order approximation

$$y(x) \sim y_{\text{outer}} = \mathrm{e}^{1-x}. \tag{3.4}$$

In reducing the procedure to solving a first-order equation (3.3), we could not impose both boundary conditions. Indeed, the above approximation does not satisfy the boundary of $y(0) = 0$, and in fact, we see that $y_{\text{outer}} \sim \mathrm{e}$ as $x \to 0$. A plot of the outer solution is shown in fig. 3.1.

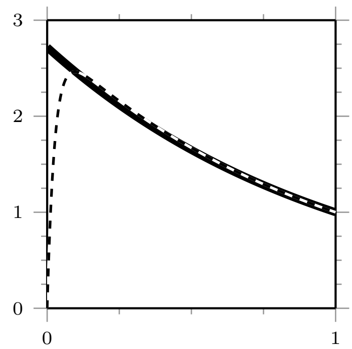

Figure 3.1: Leading-order outer solution (thick) vs. numerical solution with $\epsilon = 0.05$ (dashed).

Inner solution. The solution is to introduce a boundary layer near $x = 0$. The layer is assumed to be of order $\delta \ll 1$, and we perform a re-scaling of the coordinates using

$$x = \delta X,$$

assuming that $X = \mathcal{O}(1)$ as $\epsilon \to 0$. The inner solution is written as $y(\delta X) = Y(X)$, and the differential equation (3.1) changes to

$$\frac{\epsilon}{\delta^2} Y_{XX} + \frac{2}{\delta} Y_X + 2Y = 0. \tag{3.5}$$

The dominant balance is established from the first two terms and hence we select $\delta = \epsilon$. Thus, the resultant inner problem is to determine

$$\begin{gathered} Y'' + 2Y' + 2\epsilon Y = 0, \\ Y(0) = 0, \end{gathered} \tag{3.6}$$

and subject to appropriate conditions that ensure the inner solution matches smoothly with the outer solution. Again solving the leading-order equation yields

$$Y_0(X) = \frac{C_1}{2}\left(1 - \mathrm{e}^{-2X}\right) = C_1 \mathrm{e}^{-X} \sinh(X), \tag{3.7}$$

after imposing the boundary condition $Y_0(0) = 0$. This leaves a single constant of integration, C_1. In order for the leading-order inner solution to match the leading-order outer solution, we impose the *Prandtl matching condition* of

$$\lim_{X \to \infty} Y_0(X) = \lim_{x \to 0} y_0(x).$$

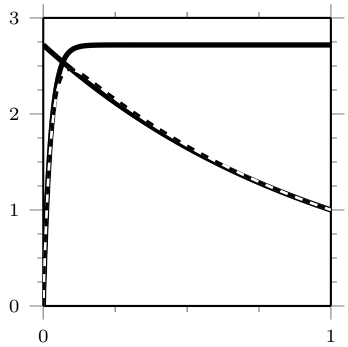

Figure 3.2: Leading-order outer solution (thick) and leading-order inner solution (thick) vs. numerical solution with $\epsilon = 0.05$ (dashed).

From (3.7) and (3.4), we see that $C_1 = 2\mathrm{e}$. A comparison of the leading-order inner and outer solutions, as compared to the full solution is shown in fig. 3.2.

It is possible to construct a uniformly valid composite solution by adding the inner and outer solutions and subtracting the overlap. This yields

$$y_{\text{unif}} \sim \mathrm{e}\Big(1 - \mathrm{e}^{2x/\epsilon}\Big) + \mathrm{e}^{1-x} - \mathrm{e}, \tag{3.8}$$

and this approximation is shown in fig. 3.3. A larger graphic is shown in fig. 3.4.

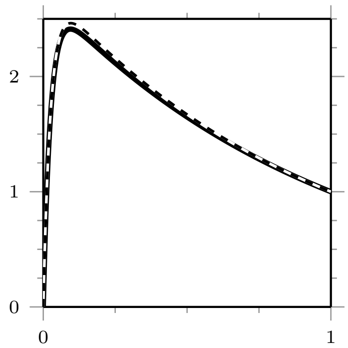

Figure 3.3: Leading-order composition solution (thick) and vs. numerical solution with $\epsilon = 0.05$ (dashed).

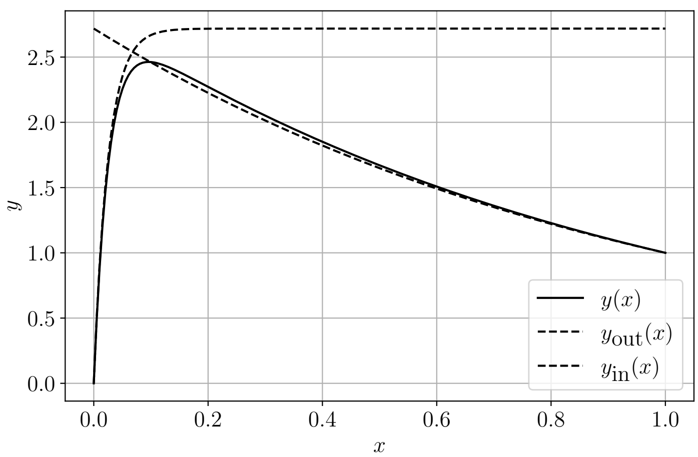

Figure 3.4: Leading-order inner and outer asymptotic approximations (dashed) compared with the full numerical solution (solid) at $\epsilon = 0.05$.

Let us proceed to higher-order in the above problem. From (2.1), at next order in the outer region, we seek to solve $y_1' + y_1 = -y_0''/2$ subject to $y_1(0) = 0$. Similarly, from (3.6), at next order in the inner region, we see to solve $Y_1'' + 2Y_1' = -2Y_0$ subject to $Y_1(0) = 0$. Together with our previous leading-order approximations, this yields

$$y(x) \sim \left[\mathrm{e}^{1-x}\right] + \epsilon\left[\frac{1}{2}(1-x)\mathrm{e}^{1-x}\right], \tag{3.9}$$

$$Y(X) \sim \left[\mathrm{e}(1-\mathrm{e}^{-2X})\right] + \epsilon\left[B(1-\mathrm{e}^{-2X}) - X\mathrm{e}(1-\mathrm{e}^{-2X})\right], \tag{3.10}$$

where B is now the unknown constant to be determined by matching. Already at $\mathcal{O}(\epsilon)$, notice that the matching of inner and outer asymptotic expansions beyond the leading-order term can be quite complicated. The main difficulty is that as $x \to 0$ from the outer region and as $X \to \infty$ from the inner region, the asymptotic expansion loses its well-ordered nature. Both outer and inner expansions are not uniformly valid as $\epsilon \to 0$. Thus when matching, both terms at $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ in one region may contribute to terms at $\mathcal{O}(\epsilon)$ in another.

One way to proceed is to introduce an intermediate scaling. We introduce an intermediate variable, η defined by

$$x = \eta\epsilon^{\alpha} = \epsilon X \quad \text{where } 0 < \alpha < 1.$$

We then take $\epsilon \to 0$ with η fixed and this causes $x \to 0$ and $X \to \infty$. In ordering the following terms, it is useful to consider a specific value of, say, $\alpha = 1/2$. From (3.9) we get

$$y_{\text{out}} = \left[\mathrm{e} - \underbrace{\mathrm{e}x}_{(A)} + \frac{\mathrm{e}x^2}{2} + \mathcal{O}(x^3)\right] + \epsilon\frac{\mathrm{e}}{2}\left[1 - 2x + \frac{3x^2}{2} + \mathcal{O}(x^3)\right] + \mathcal{O}(\epsilon^2). \tag{3.11}$$

We then substitute $x = \epsilon^{\alpha}\eta$ and obtain

$$y_{\text{out}} \sim \left[\mathrm{e}\right] + \epsilon^{\alpha}\left[-\mathrm{e}\eta\right] + \epsilon\left[\frac{\mathrm{e}}{2}\right] + \mathcal{O}(\mathrm{e}^{2\alpha}, \epsilon^{1+\alpha}). \tag{3.12}$$

From the inner solution, (3.10), we note that all terms like e^{-2X} are exponentially small in the intermediate region, and will thus be negligible to all algebraic orders of ϵ. We may then consider $Y \simeq \mathrm{e} + \epsilon(B - X\epsilon)$. Thus

$$Y_{\text{in}} \sim \left[\mathrm{e}\right] + \epsilon^{\alpha}\left[\underbrace{-\mathrm{e}\eta}_{(A)}\right] + \epsilon\left[B\right]. \tag{3.13}$$

Comparing (3.12) to (3.13), we see that

$$B = \frac{\mathrm{e}}{2}.$$

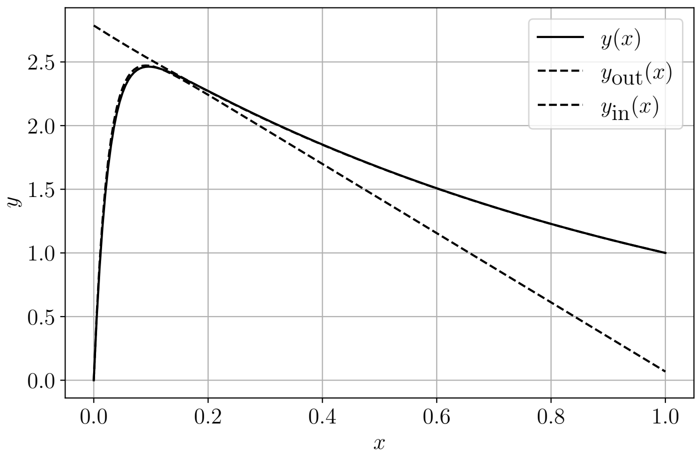

Figure 3.5: Exact (solid) vs. two-term inner and outer asymptotic approximations (dashed) for $\epsilon = 0.05$.

The two-term inner and outer expansions for the solution with $\epsilon = 0.05$ is shown in fig. 3.5.

In studying this intermediate matching procedure, we remark two issues. First, as inner and outer solutions are matched, it is normal for terms in one asymptotic order to jump to another. For instance, the term marked (A) in (3.11) is part of the leading-order approximation y_0 but is in the $\mathcal{O}(\epsilon)$ approximation of Y_1 in (3.13). This potentially complex re-arrangement of orders is a staple of matched asymptotics problems. Our second remark is that there are further terms in (3.12) that cannot be matched to (3.13). These will involve higher-order terms of Y.

3.3 VAN DYKE'S MATCHING RULE

As we have seen, matching via an intermediate coordinate is a tedious procedure. In his classical text, Van Dyke [1975] introduced a heuristic now referred to as *Van Dyke's matching principle* or *Van Dyke's matching rule*. To introduce the rule, we develop the following notation following Hinch [1991].

First, we define

$$\begin{aligned}\mathbf{Ou}_P(y) &= \text{outer solution } (x \text{ fixed}, \epsilon \to 0) \text{ retaining } P+1 \text{ terms} \\ &= \sum_{n=0}^{P} \epsilon^n y_n(x),\end{aligned}$$

and $\mathbf{Ou}_P(f)$ is the $P+1$-term outer approximation. Similarly,

$$\begin{aligned}\mathbf{In}_Q(y) &= \text{inner solution } (X \text{ fixed}, \epsilon \to 0) \text{ retaining } Q+1 \text{ terms} \\ &= \sum_{n=0}^{Q} \epsilon^n Y_n(X),\end{aligned}$$

and hence is the $Q+1$-term inner approximation.

Van Dyke's matching rule is then:

$$\mathbf{In}_Q(\mathbf{Ou}_P y) = \mathbf{Ou}_P(\mathbf{In}_Q y).$$

In words:

$$\begin{aligned} &\text{The } (Q+1)\text{-term inner expansion of } \Big[\text{ the } (P+1)\text{-term outer expansion}\Big] \\ &= \text{the } (P+1)\text{-term outer expansion of } \Big[\text{ the } (Q+1)\text{-term inner expansion}\Big]. \end{aligned}$$

We demonstrate the Van Dyke matching rule with (3.9) and (3.10). Beginning firstly with $P = Q = 0$, we have

$$\begin{aligned} \mathbf{In}_0\mathbf{Ou}_0 y &= \mathbf{In}_0[y_0(x)] \\ &= \mathbf{In}_0[\mathrm{e}^{1-\epsilon X}] && \text{(re-write in inner coordinates)} \\ &= \mathbf{In}_0[\mathrm{e}(1-\epsilon X+\ldots)] && \text{(re-expand with } X \text{ fixed)} \\ &= \mathrm{e}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{Ou}_0\mathbf{In}_0 y &= \mathbf{Ou}_0[Y_0(X)] \\ &= \mathbf{Ou}_0[\mathrm{e}(1-\mathrm{e}^{-x/\epsilon})] && \text{(re-write in outer coordinates)} \\ &= \mathrm{e}, && \text{(re-expand to one term)} \end{aligned}$$

which verifies our choice of coefficient in earlier applying Prandtl's matching rule. Proceeding to one order higher, we choose to apply Van Dyke's rule at $P = Q = 1$. Then

$$\begin{aligned} \mathbf{In}_1\mathbf{Ou}_1 y &= \mathbf{In}_1\{y_0(x) + \epsilon y_1(x)\} \\ &= \mathbf{In}_1\left\{\left[\mathrm{e}^{1-\epsilon X}\right] + \epsilon\left[\frac{1}{2}(1-\epsilon X)\mathrm{e}^{1-\epsilon X}\right]\right\} && \text{(re-write in inner coordinates)} \\ &= \mathrm{e} + \epsilon\left[\frac{\mathrm{e}}{2} - \mathrm{e}X\right]. && \text{(re-expand to two terms)} \end{aligned}$$

While we have from the other side,

$$\begin{aligned} \mathbf{Ou}_1\mathbf{In}_1 y &= \mathbf{Ou}_1[Y_0(X) + \epsilon Y_1(X)] \\ &= \mathbf{Ou}_1\left\{\left[\mathrm{e}(1-\mathrm{e}^{-2x/\epsilon})\right] + \epsilon\left[B(1-\mathrm{e}^{-2x/\epsilon}) - \frac{x}{\epsilon}\mathrm{e}(1-\mathrm{e}^{-2x/\epsilon})\right]\right\} \\ & \qquad\qquad\qquad\qquad \text{(re-write in outer coordinates)} \\ &= \left[\mathrm{e} - x\mathrm{e}\right] + \epsilon\left[B\right] && \text{(re-expand to two terms)} \\ &= \left[\mathrm{e}\right] + \epsilon\left[B - \mathrm{e}X\right] && \text{(re-write in terms of } X\text{)} \end{aligned}$$

Comparing the two final lines above, we again see the required match of $B = \frac{\mathrm{e}}{2}$. In general, it can be verified that Van Dyke's matching can also be applied in a diagonal fashion, with *e.g.* $P = 0$ and $Q = 1$.

3.3.1 *The failure of Van Dyke's rule*

A great deal has been written about the potential failure of Van Dyke's matching principle. There are extensive discussions in Van Dyke [1975, Note 4, p. 220], Hinch [1991, p. 72], and Eckhaus [1994]. The restriction formulated by Van Dyke [1975, p. 221] is *"Don't cut between logarithms"*. The majority of cases illustrating a potential failure of the matching rule involve the ambiguity of treating terms of $\mathcal{O}(\epsilon \log \epsilon)$ in comparison with terms like $\mathcal{O}(\epsilon)$. Van Dyke's maxim refers to the fact that it may be necessary to consider these two orders as equivalent in order to perform matching.[1]

[1]As Hinch [1991, p. 72] notes: *"When applying Van Dyke's rule, good advice is to match only at a break where the power of ϵ changes, if that is possible."*

Apart from a subset of such problems, Van Dyke's rule seems to work as intended for most problems (that is certainly the case for all the problems presented in this book); for problems where there is a "failure", it is often possible to re-interpret Van Dyke's rule—such as in the case of logarithmic orders.[2]

[2]As noted by Eckhaus [1994]: *"...the discussion is, to some extent, academic: any intelligent practitioner of applied analysis will find [there] way to correct matching in a given problem, no matter what [their] convictions are..."*.

3.4 BOUNDARY LAYER LOCATION AND ITS PROPERTIES

In simple problems, such as the case of (3.1) it is possible to predict the boundary layer location *a priori*. Consider the general second-order linear example,

$$\epsilon y'' + p(x)y' + q(x)y = 0,$$

with $y(0) = A$ and $y(1) = B$. As usual, we shall consider $\epsilon > 0$ and $\epsilon \to 0$. If there exists a boundary layer, then it must appear in a location where the derivatives are large, and hence dominant balance indicates

$$\epsilon y'' \sim -p(x)y'.$$

Four possibilities are sketched in the figures [—]. For the case of the boundary layer on the left: (i) $y'' > 0$ and $y' < 0$; (ii) $y'' < 0$ and $y' > 0$, the curvature and gradient must possess different signs in general, hence this corresponds to situations where $p > 0$. The problem (3.1) is such an example. For the case of the boundary layer on the right: (iii) $y'' > 0$ and $y' > 0$; and (iv) $y'' < 0$ and $y' < 0$, and thus the curvature and gradient possess the same sign. Hence this corresponds to $p < 0$. If there exists a point in the domain where $p = 0$, then internal boundary layers are possible. In general, however, such geometric arguments are not possible, certainly for the case of nonlinear differential equations where boundary layers may occur in a spontaneous fashion (unpredicted by the coefficients of the differential equation).

3.5 MATCHING NEAR SINGULARITIES

The method of matched asymptotics will form an important set of tools for exponential asymptotics, but we will not often encounter them in the context of two-point boundary-value problems. Instead, we will need the technique in order to study the outer- and inner-asymptotic expansions near a general singularity.

We provide an example of this procedure. Consider[3]

[3]Compare to eqn (2.1) of Akylas and Yang [1995]

$$\epsilon^2 u'' - \epsilon u^2 + u = \operatorname{sech}(x). \tag{3.14}$$

The boundary conditions are not important for the discussion, but they can be considered to be $u, u' \to 0$ as $x \to -\infty$. In the limit $\epsilon \to 0$, we first study the regular asymptotic expansion in powers of ϵ, with

$$u \sim u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \dots \tag{3.15}$$

Solving for the first few orders yields

$$u_0(x) = \operatorname{sech} x, \tag{3.16a}$$
$$u_1(x) = \operatorname{sech}^2 x, \tag{3.16b}$$
$$u_2(x) = 3\operatorname{sech}^3 x - \operatorname{sech} x \tanh^2 x. \tag{3.16c}$$

We note that the leading-order approximation, $\operatorname{sech} x$, is singular[4] at $x = \left(\frac{2n+1}{2}\right)\pi\mathrm{i}$ for $n = 0, \pm 1, \pm 2, \dots$. In fact, we notice that this singularity is present in the first three orders in (3.16). Indeed, since all subsequent orders depend on differentiation of the previous, the singularities are these points along the imaginary axis are expected to be present to all orders, and we furthermore argue that the power of the singularity will grow as the order increases.

[4] $\operatorname{sech} x = \dfrac{2}{\mathrm{e}^x + \mathrm{e}^{-x}}$

Let us consider the behaviour of the asymptotic solution as $x \to \pi\mathrm{i}/2$. Notice that

$$\operatorname{sech} x \sim -\frac{\mathrm{i}}{x - \pi\mathrm{i}/2} + \mathcal{O}(x - \pi\mathrm{i}/2).$$

We ask the question of how close x must be to $\pi\mathrm{i}/2$ before the second term the asymptotic expansion (3.15) becomes of the same order as first. Equating $u_0 = \mathcal{O}(\epsilon u_1)$, we see that

$$\frac{1}{x - \pi\mathrm{i}/2} = \mathcal{O}\left(\frac{\epsilon}{(x - \pi\mathrm{i}/2)^2}\right),$$

and hence it is when $x - \pi\mathrm{i}/2 = \mathcal{O}(\epsilon)$, that the expansion (3.15). We then say that the boundary layer is of size $\mathcal{O}(\epsilon)$. Within the layer, it can be verified that all the terms of (3.15) re-combine to be of the same order of magnitude. This rationalisation of the breakdown of the asymptotic expansion is important enough to highlight:

Remark 3.1 (Re-ordering of terms in the boundary layer)
In general, the location of a boundary layer near a singularity can be predicted by examining where terms of an asymptotic expansion become of equal size.

We wish to develop the inner solution within the boundary layer. Based on (3.16a), we intuit that in this region, $u = \mathcal{O}(1/\epsilon)$. Thus we re-scale the independent and dependent variables as

$$x = \frac{\pi\mathrm{i}}{2} + \epsilon X \quad \text{and} \quad u(x) = \frac{U(X)}{\epsilon}, \tag{3.17}$$

with X and U assumed to be $\mathcal{O}(1)$ in the boundary layer. The differential equation (3.14) gives

$$U'' - U^2 + U = \epsilon \operatorname{sech}\left(\frac{\pi\mathrm{i}}{2} + \epsilon X\right) = -\frac{\mathrm{i}}{X} + \epsilon^2\left[\frac{\mathrm{i}X}{6}\right] + \mathcal{O}(\epsilon^4). \tag{3.18}$$

We shall provide an example of the asymptotic matching between inner and outer regions by examining the leading-order problem:

$$U_0'' - U_0^2 + U_0 = -\frac{\mathrm{i}}{X}. \tag{3.19}$$

The nonlinear differential equation does not have an obvious closed-form solution, but we seek to determine its asymptotic behaviour in the limit that the outer region is approached, or $X \to \infty$. The limit of $X \to \infty$ is a regular point and the solution can be expanded into a Taylor series,

$$U_0(X) = \sum_{n=0}^{\infty} \frac{A_n}{X^n} \qquad \text{as } X \to \infty.$$

Substitution into the differential equation produces the following recurrence relations:

$$\begin{aligned} -A_0^2 + A_0 &= 0 \\ -2A_0A_1 + A_1 &= -\mathrm{i}, \\ (n-2)(n-1)A_{n-2} - \sum_{m=0}^{n} A_m A_{n-m} + A_n &= 0, \qquad n \geq 2. \end{aligned} \tag{3.20}$$

Notice in the outer region, the leading-order solution is $u_0 = \operatorname{sech} x \sim -\mathrm{i}/(\epsilon X)$. Since U is scaled according to (3.17), then we expect for $A_0 = 0$. Solving for the first few orders yields $A_0 = 0$, $A_1 = -\mathrm{i}$, $A_2 = -1$, $A_3 = 4\mathrm{i}$, and thus,

$$U_0 = -\frac{\mathrm{i}}{X} - \frac{1}{X^2} + \frac{4\mathrm{i}}{X^3} + \mathcal{O}\left(\frac{1}{X^4}\right), \tag{3.21}$$

represents the outer limit of the leading-order inner solution.

We finally demonstrate how the outer solutions, (3.16) relate to the leading-order inner solution (3.21) according to the Van-Dyke matching rule. With only a single order of the inner solution, we thus consider the Van-Dyke matching rule with $Q = 0$.

First with $P = 0$ and $Q = 0$,

$$\begin{aligned} \mathbf{In}_0\mathbf{Ou}_0 y &= \mathbf{In}_0[u_0(x)] \\ &= \mathbf{In}_0\left[\operatorname{sech}\left(\frac{\pi\mathrm{i}}{2} + \epsilon X\right)\right] && \text{(re-write in inner coordinates)} \\ &= -\frac{\mathrm{i}}{\epsilon X} && \text{(re-expand to one term).} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{Ou}_0\mathbf{In}_0 u &= \mathbf{Ou}_0\left[\frac{U_0(X)}{\epsilon}\right] \\ &= \mathbf{Ou}_0\left[\frac{1}{\epsilon}\sum_{m=0}^{\infty}\frac{A_m}{(x/\epsilon)^m}\right] && \text{(re-write in outer coordinates)} \\ &= -\frac{\mathrm{i}}{x} && \text{(re-expand to one term),} \end{aligned}$$

and hence Van-Dyke matches to leading order. The same procedure can be carried out to verify that matching holes for $(P, Q) = (1, 0)$ and $(P, Q) = (2, 0)$ and so forth.

3.6 FURTHER REFERENCES

The golden age of matched asymptotics was the 1950s [Holmes, 2012] and two classic and excellent references from that era that significantly develop the theory in a form familiar to our presentation is Van Dyke [1975] and Cole [1968]. The latter has been updated into Kevorkian and Cole [2013]. The modern treatments by Holmes [2012] and Hinch [1991] are also excellent. As always Bender and Orszag [1999] provide a compact collection of problems.

Because of its importance in singular perturbation theory and in connection to the development of aerodynamics, much has been written about the history of matched asymptotic expansions and boundary layer theory. The reviews by Cole [1994], Van Dyke [1994], O'Malley [2014], O'Malley Jr [2010] discuss the fascinating history.

3.7 EXERCISES