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Draft chapter last generated 2024-12-02; P.H. Trinh

In these first chapters, we review the essential details of asymptotic analysis. For the most part, we assume that the reader is familiar with a typical one-semester course on asymptotics or perturbation theory—the purpose of this review is to largely serve as a refresher and to provide a reference point to key terminology and definitions.

To begin, we motivate the notion of an asymptotic expansion. Consider the solution of the following boundary-value problem for a real-valued function $y = y(x)$:

$$\begin{aligned} \epsilon y'' + 2y' + 2y = 0, \qquad 0 < x < 1 \\ y(0) = 0 \quad \text{and} \quad y(1) = 1, \end{aligned} \tag{2.1}$$

where $\epsilon > 0$ is a small parameter. We may alternatively write $y = y(x; \epsilon)$ to explicitly note the ϵ dependence. The boundary-value problem can be numerically computed using standard techniques.[1] Three numerical solutions are shown in fig. 2.1 at decreasing values of $\epsilon = 0.3$, 0.1, and 0.01.

[1] We have used MATLAB's BVP4C function to compute the solutions in the image.

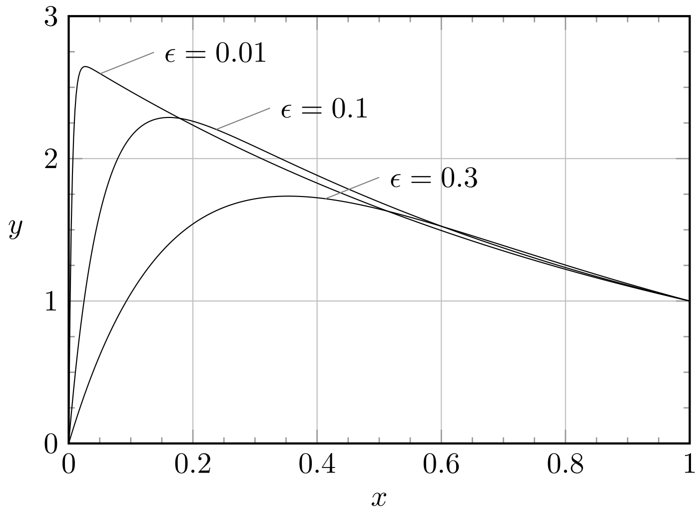

Figure 2.1: Solutions of (2.1) for three small values of ϵ.

Observe that as ϵ tends to zero, the solution of the boundary-value-problem seems to approaches a configuration with a well-defined structure. In particular, away from $x = 0$, the profile approaches a gently sloped curve. Our desire is to produce a systematic approximation procedure that describes this limit, writing, *e.g.*

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots \tag{2.2}$$

Thus the solution is approximated by series expansion in powers of the small parameter, ϵ. We expect that the determination of the individual terms of the sum (2.2) will be easier than attempting to determine the full un-approximated solution. At the same time, we see from fig. 2.1

that for small values of ϵ, the solution exhibits a *boundary layer* near $x = 0$—that is, a region where it rapidly changes in value or gradient. The goal of an asymptotic procedure should be to develop accurate approximations of the solutions in both the regions near $x = 0$ and away from $x = 0$.

2.1 ASYMPTOTIC EQUIVALENCE AND NOTATION

Our previous example of the approximation (2.2) is an example of a *parametric expansion* for y, where the solution can be regarded as a function of two variables x and ϵ, with the latter considered to be a small number. In this book, we will generally work with functions of this form. Below, for simplicity of presentation, we state the definitions for a function such as $f(z;\epsilon)$ where z is assumed to be fixed through the limiting process of $\epsilon \to 0$.[2] Hence below we write, *e.g.* $f(\epsilon)$ and $g(\epsilon)$, and temporarily drop dependence on other variables.

[2] It is typical to first introduce asymptotic expansions in the context of functions, say $f(x)$ as $x \to 0$ or as $x \to \infty$ and to later present the case of a parametric function, $f(x;\epsilon)$ determined by a problem with an additional parameter. In such cases, the definitions are equivalent to setting, *e.g.* $x = \epsilon$ or $x = 1/\epsilon$. This is the procedure of authors such as Hinch [1991].

Definition 2.1 (Asymptotic equivalence)
Functions $f(\epsilon)$ and $g(\epsilon)$ are asymptotically equivalent[3] in the limit $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = 1.$$

This relationship is denoted $f(\epsilon) \sim g(\epsilon)$.

[3] We often say that f "behaves like" g or f "twiddles" g. In the mathematical sciences, there is further ambiguity with some researchers writing, for instance, that $\epsilon^2 \sim 2\epsilon^2$ as $\epsilon \to 0$. Thus, for some researchers, the $\sim$ relationship is specified up to a constant pre-factor.

Note that asymptotic equivalence does not necessarily mean that f and g have the exact same functional form in the limit; only that they have the same rate of growth or decay. Examples of asymptotically equivalent functions in the limit $\epsilon \to 0$ are

$$\frac{1}{\epsilon} \sim 1 + \frac{1}{\epsilon}, \qquad \sin(\epsilon) \sim \epsilon, \qquad \mathrm{e}^{\epsilon} \sim 1 + \epsilon + \frac{1}{2}\epsilon^2, \qquad \tan(\tfrac{\pi}{2} - \epsilon) \sim \frac{1}{\epsilon}.$$

Asymptotic equivalence is also commonly written in the limit a variable tends to infinity. For instance, we write

$$x^2 \sim \frac{x^4 + 2x^2 + 1}{x^2 + \sin(x)} \qquad \text{as } x \to \infty,$$

and the definition 2.1 is extended a similar way to $x \to \infty$, or alternatively by setting $\epsilon = 1/x \to 0$.

Definition 2.2 (Big $\mathcal{O}$ notation)
The 'big $\mathcal{O}$' is used to state that a function $f(\epsilon)$ is at most of order $g(\epsilon)$ as $\epsilon \to 0$. That is $f = \mathcal{O}(g)$ as $\epsilon \to 0$ if

$$\left|\frac{f(\epsilon)}{g(\epsilon)}\right| < M, \qquad \text{as } \epsilon \to 0,$$

where M is an arbitrary real constant.

Definition 2.3 (Little $\mathcal{O}$, $\gg$ and $\ll$ notation)
The 'little $\mathcal{O}$' is used to state that a function, say $f(\epsilon)$ is of smaller magnitude than another, say $g(\epsilon)$ as $\epsilon \to 0$. Symbolically we shall write $f = \mathcal{O}(g)$ as $\epsilon \to 0$ if

$$\left| \frac{f(\epsilon)}{g(\epsilon)} \right| \to 0, \qquad \text{as } \epsilon \to 0.$$

In this case, we also write $f \ll g$ and say that "f is much less than g". Or equivalently, we write $g \gg f$ and we say that "g is much greater than f".

2.2 REGULAR AND SINGULAR PERTURBATION PROBLEMS

In general, the goal of asymptotic analysis is to characterise the solution of a problem (*e.g.* an algebraic equation, a differential equation, a recurrence relation, etc.) in the limit that certain parameters tend to zero or infinity. A *singular* perturbative problem is one in which the solution at the limit point, say $\epsilon = 0$, differs fundamentally from the solution in the limit $\epsilon \to 0$. A problem is *regularly perturbed* in the limit $\epsilon \to 0$ if it is not singular. This terminology is best demonstrated with examples.

Example 2.1 (A regular algebraic problem)
Consider the roots of $x^2 - x + \epsilon = 0$ where ϵ is small. In the limit $\epsilon \to 0$ ($\epsilon \neq 0$), then we have a quadratic with two roots. If we now set $\epsilon = 0$ then we again have a quadratic with two roots. Hence this is a regular problem; the introduction of a small ϵ only serves to shift each of the two roots by a small amount.

Example 2.2 (A singular algebraic problem)
Consider now $\epsilon x^2 + x + 1 = 0$. In the limit $\epsilon \to 0$ ($\epsilon \neq 0$) we have a quadratic with two roots, but if we set $\epsilon = 0$ then the problem has only one root, with $x = -1$. This is a singularly perturbed problem. In the limit of $\epsilon \to 0$, one of the two roots tends to infinity.

Example 2.3 (A singular differential equation)
The boundary-value problem in (2.1) is an example of a singular differential equation. Setting $\epsilon = 0$ results in a first-order differential equation and hence only one boundary condition is typically required for a unique solution. However for any $\epsilon \neq 0$, the differential equation is of second order and hence requires two boundary conditions. Thus problem for $\epsilon = 0$ differs fundamentally from the problem for $\epsilon \neq 0$.

2.3 ASYMPTOTIC EXPANSIONS FOR ALGEBRAIC EQUATIONS

We demonstrate how asymptotic analysis is used to develop approximations to the two examples above.

Example 2.4 (The expansion method for a regular problem)
Consider the *regularly* perturbed quadratic equation

$$x^2 - x + \epsilon = 0, \qquad \text{as } \epsilon \to 0, \tag{2.3}$$

introduced in example 2.1. The exact solution of this quadratic is given by

$$x = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-4\epsilon}.$$

As $\epsilon \to 0$, our two solutions tend to 0 and 1, which are the unperturbed roots. By expanding the exact solution in a series of increasing powers[4] of ϵ we intuit that the two roots can be expanded into a series in integer powers of ϵ. Thus, we shall write

[4] By Taylor expansions, note that $(1-4\epsilon)^{1/2} = 1 - 2\epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3)$.

$$x = \sum_{n=0}^{\infty} \epsilon^n x_n = x_0 + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \mathcal{O}(\epsilon^4), \qquad (2.4)$$

and attempt to solve for x_0, x_1, ... We substitute this into (2.3) and group terms for each order of ϵ we have

$$(x_0^2 - x_0) + (2x_0x_1 - x_1 + 1)\epsilon + (x_1^2 + 2x_0x_2 - x_2)\epsilon^2 + (2x_0x_3 + 2x_1x_2 - x_2)\epsilon^3 + \mathcal{O}(\epsilon^4).$$

Equating each respective order of ϵ to zero then yields

$$\begin{aligned}
\epsilon^0 &: \quad x_0^2 - x_0 = 0 && \Rightarrow \quad x_0 = 0 \text{ or } x_0 = 1,\\
\epsilon^1 &: \quad 2x_0x_1 - x_1 + 1 = 0 && \Rightarrow \quad x_1 = 1 \text{ or } x_1 = -1,\\
\epsilon^2 &: \quad x_1^2 + 2x_0x_2 - x_2 = 0 && \Rightarrow \quad x_2 = 1 \text{ or } x_2 = -1,\\
\epsilon^3 &: \quad 2x_0x_3 + 2x_1x_2 - x_2 = 0 && \Rightarrow \quad x_3 = 2 \text{ or } x_3 = -2.
\end{aligned}$$

The procedure can be continued *ad infinitum*. Notice that our values of x_0 are the unperturbed roots for the equation. Using the above we can conclude that the estimates for the solutions are thus

$$x = \epsilon + \epsilon^2 + 2\epsilon^3 + \mathcal{O}(\epsilon^4),$$
$$x = 1 - \epsilon - \epsilon^2 - 2\epsilon^3 + \mathcal{O}(\epsilon^4).$$

The graph showing the error of the convergent series approximation for the root near $x = 1$ is shown in fig. 2.2.

Why choose integer powers of ϵ? In Example 2.4, we have taken the expansion of the roots in integer powers of ϵ. In many cases, the appropriate expansion can be observed by examining the remainder; for instancing, substituting $x \approx x_0 = -1$ into $\epsilon x^2 + x + 1 = 0$, we observe an unbalanced term of $\mathcal{O}(\epsilon)$ on the left hand-side. For a more developed explanation of this concept see Hinch [1991] Section 1.3.

Example 2.5 (The expansion method for a singular problem and the method
The expansion in Example 2.1 was straightforward. Consider now the singular problem mentioned of Example 2.2, where we seek the roots of

$$\epsilon x^2 + x + 1 = 0, \qquad \text{as } \epsilon \to 0. \qquad (2.5)$$

If we naively set $\epsilon = 0$ in (2.5), we obtain only a single root near $x \sim -1$. Again, we may expand x as in (2.4) and solve for each term; this yields the asymptotic expansion of $x = -1 - \epsilon - 2\epsilon^2 + \ldots$

It remains to study the other root. Notice that if $|x|$ is large, then the assumption that ϵx^2 is small may no longer be true. Thus as $\epsilon \to 0$, we conjecture that the secondary root has $|x| \to \infty$. We thus propose the rescaling

$$x = \frac{X}{\delta} \qquad (2.6)$$

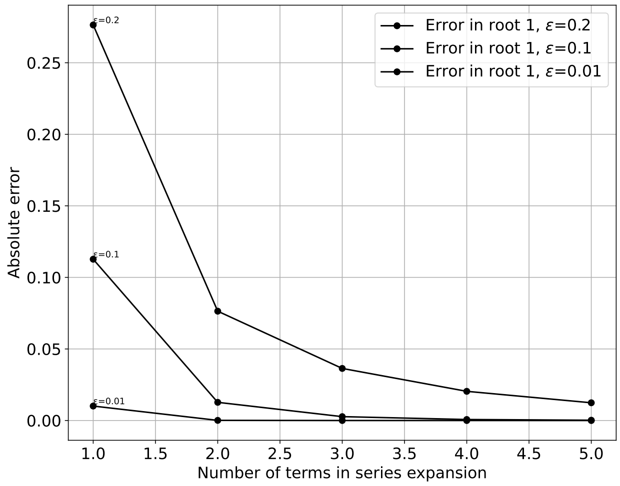

Figure 2.2: Error of the first root, approximated as $x = 1 - \epsilon - \epsilon^2 + \ldots$ corresponding to the quadratic $x^2 - x + \epsilon = 0$.

where $\delta \to 0$ as $\epsilon \to 0$. We seek to choose δ appropriately so that the re-scaled root, X, remains $\mathcal{O}(1)$ as $\epsilon \to 0$. In essence, this is a re-scaling of the coordinate, x, so that under the new coordinate system, the desired root is fixed as $\epsilon \to 0$. Now equation (2.5) becomes

$$\underbrace{\frac{\epsilon}{\delta^2}X^2}_{①} + \underbrace{\frac{1}{\delta}X}_{②} - \underbrace{1}_{③} = 0. \tag{2.7}$$

The *method of dominant balance* identifies which terms of (2.7) can be taken to be negligible, and which terms are involved in establishing the leading-order equality. There are four possible dominant balances.[5] It is sensible to include ① (since this term is necessary in order to involve the quadratic factor). And since x is large, then we expect ② $\gg$ ③. Let us thus posit the leading-order balance of ① $\sim$ ②. Then

[5] We can have the four possibilities of [1]∼[2]; [2]∼[3]; [1]∼[3]; [1]∼[2]∼[3].

$$\frac{\epsilon}{\delta^2}X^2 \sim -\frac{1}{\delta}X,$$

and we thus choose $\delta = \epsilon$, which yields the leading-order approximation of $X \sim -1$. Notice with this choice of δ, then ② $= \mathcal{O}(X/\epsilon)$ while ③ is $\mathcal{O}(1)$. This confirms *a posteriori* that

$$① \sim ② \gg ③,$$

as we originally posited. The remaining three possible dominant balances can be discounted by reaching contradictions. We may now return to (2.7), with $\delta = \epsilon$, and study

$$X^2 + X - \epsilon = 0. \tag{2.8}$$

This is now a regular perturbation problem. We may then pose the expansion $X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \ldots$ and solve for the asymptotic approximation term-by-term. The first three orders yields $X \sim -1 -$

$\epsilon + -2\epsilon^2$. Thus the two roots of (2.5) are

$$x = -1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3),$$
$$x = \frac{1}{\epsilon}\left[-1 - \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3)\right].$$

It is observed that both series approximations are divergent. For instance, the coefficient of ϵ^8, ϵ^9, and ϵ^{10} for both series are respectively, $\mp 1430, \mp 4862, \mp 16796$ with the negative sign for the root near -1. A graph of the errors associated to using the approximation near the root near $x \sim -1$ is shown in fig. 2.3. The divergence of the singular approximation can be remarked in the figure for the series with $\epsilon = 0.3$.

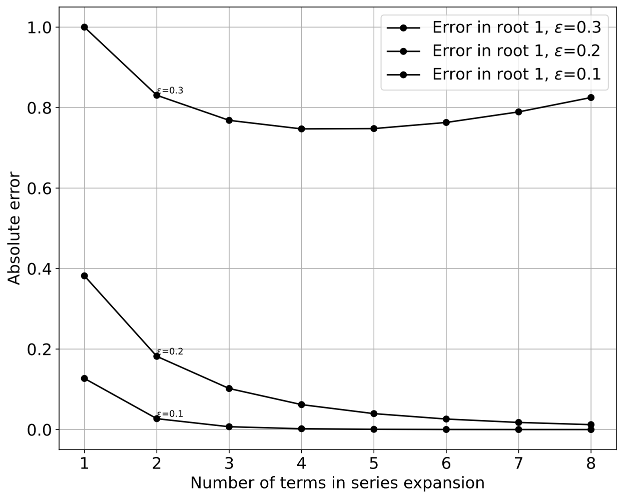

Figure 2.3: Error of the first root, approximated as $x = -1 - \epsilon - 2\epsilon^2 + \ldots$ corresponding to the singular quadratic $\epsilon x^2 + x + 1 = 0$. The divergence is remarked for the curve corresponding to $\epsilon = 0.3$.

2.3.1 Newton graphs

There is a graphical procedure by which the method of dominant balance can be visualised. Instead of the re-scaling (2.6), let us set $x = X/\epsilon^\alpha$ for some α to be determined. Then (2.5) yields

$$\epsilon^{1-2\alpha} X^2 + \epsilon^{-\alpha} X - \epsilon^0 \cdot 1 = 0. \quad (2.9)$$

We then plot a graph with the three ϵ exponents of $\{1 - 2\alpha, -\alpha, 0\}$. Intersections between the three curves correspond to possible dominant balances. In order to be a consistent dominant balance, the remaining curves must lie higher than the intersection. From the graphic, we thus see that the two possible dominant balances are $\alpha = 0$ and $\alpha = -1$.

2.4 ASYMPTOTIC EXPANSIONS

In the previous section, we provided two examples of an asymptotic approximation for the solution to an algebraic equation. We wish to define more rigorously this notion of an asymptotic approximation. For ease of notation in the following definitions, we shall index sequences using the non-negative integers, $n \in \mathbb{Z}^* = \{0, 1, 2, \ldots\}$, and below, we always consider the limit of $\epsilon \to 0$.

Typically, a "complete" asymptotic approximation is associated with a countably infinite number of terms indexed in the above manner. However, one can certainly adapt the below definitions to include the cases of a finite number of terms (cf. Hinch 1991).

Definition 2.1 (Asymptotic sequence)
A sequence $\{\delta_n(\epsilon)\}_{n\geq 0}$ is said to be asymptotic sequence if, for all $n > 0$, each subsequent term in the sequence is much smaller than the previous, $\delta_{n+1} \ll \delta_n$. Equivalently

$$\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0. \tag{2.10}$$

We might alternatively write this as $\delta_{n+1} = o(\delta_n)$.

Definition 2.2 (Asymptotic expansion/approximation)
Let $\{\delta_n(\epsilon)\}$ be an asymptotic sequence. Let $f_n(\epsilon) = a_n\delta_n(\epsilon)$ for given constants a_n. We say that $\sum_{n=0}^{\infty} f_n(\epsilon)$ is an *asymptotic expansion* or *asymptotic approximation* of $f(\epsilon)$ as $\epsilon \to 0$ if, for all $N > 1$, we have

$$\frac{f(\epsilon) - \sum_{n=0}^{N-1} f_n(\epsilon)}{f_{N-1}(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0. \tag{2.11}$$

Note that since

$$R_N(\epsilon) = f(\epsilon) - \sum_{n=0}^{N-1} f_n(\epsilon) \tag{2.12}$$

is the remainder[6] after an N-term expansion, the above definition of an asymptotic expansion expresses the fact that $R_N \ll f_{N-1}$, i.e. the remainder is smaller than the last term included if ϵ is sufficiently small. If the sum has this asymptotic property, then we write

[6] To be consistent with later notation, we define the remainder R_N in association with the asymptotic expansion truncated after N terms.

$$f(\epsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon).$$

Remark 2.1 (Asymptotic progressions)
Standard asymptotic expansions typically involve sequences consisting of integral or rational powers of ϵ. For instance, a common asymptotic expansion is $\sum_{n=0}^{\infty} a_n\epsilon^n$. More exotic expansions can involve logarithmic terms, such as the nested expansion $\log(1/\epsilon) + \log(\log(1/\epsilon)) + \ldots$ [cf. [?, §1.4]], or expansions where the dependences of f_n on ϵ cannot be expressed in closed form[7].

[7] An example of this occurs in studying the asymptotic expansion near the crest of a steep water waves (the Stokes wave). See Grant [1973].

Remark 2.2 (Uniqueness of asymptotic approximations)
Given the asymptotic sequence $\{\delta_n(\epsilon)\}$ used to approximate $f(\epsilon)$, then the coefficients, a_n, in the approximation $f \sim \sum a_n\delta_n(\epsilon)$ are unique.

They can be determined inductively by

$$a_k \equiv \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\epsilon)}{\delta_k(\epsilon)} \qquad \text{for } k = 1, 2, 3, ...$$

with $a_0 = \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\delta_0(\epsilon)}$. Note that two different functions may share the same asymptotic approximation if they differ by a quantity smaller than the last term in the approximation.

While we have uniqueness for a given asymptotic sequence, the function f may have many asymptotic approximations, for instance as $\epsilon \to 0$

$$\tan \epsilon \sim \epsilon + \tfrac{1}{3}\epsilon^3 + \tfrac{2}{15}\epsilon^5 \sim \sin \epsilon + \tfrac{1}{2}(\sin \epsilon)^3 + \tfrac{3}{8}(\sin \epsilon)^5,$$

are two distinct expansions to $\tan \epsilon$.

Remark 2.3 (Non-analyticity)
Let us consider the asymptotic expansions of $f(\epsilon) = \exp(\epsilon)$ and $g(\epsilon) = \exp(\epsilon) + \exp(-1/\epsilon)$ as $\epsilon \searrow 0$ ($\epsilon \to 0$ through positive values only). Then note

$$\exp(\epsilon) \sim \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \quad \text{and} \quad \exp(\epsilon) + \exp(-1/\epsilon) \sim \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!},$$

and hence f and g possess the same asymptotic expansion. This does not contradict Remark 2.2 since f and g differ by a quantity which is not analytic. They have identical asymptotic power series, but are not equal functions.

Remark 2.4 (Manipulations of asymptotic approximations)
Asymptotic expansions can be added, subtracted, multiplied and divided. They can also be integrated term-by-term with respect to ϵ resulting in the correct asymptotic expression for an integral. If the function $f(\epsilon)$ is analytic in some domain in the complex ϵ-plane, then we can differentiate the asymptotic approximation in this domain also. However we cannot safely differentiate an approximation otherwise.

Remark 2.5 (Asymptotic expansions in other limits)
In the definitions above, we have formulated asymptotic expansions in the limit $\epsilon \to 0$ since this provides a common framework for most applications in perturbation theory. These definitions can be adapted for the asymptotic expansions of *e.g.* $f(z)$ as $z \to z_0$ or $z \to \infty$. In most cases, expansions in other limits can be re-formulated as an $\epsilon \to 0$ limit through a variable transformation. Note that it is also common to develop asymptotic expansions for limiting processes over multiple variables. For instance, one can study $f(z; \epsilon)$ as $z \to \infty$ and $\epsilon \to 0$. In such cases, there may exist subtlety to the approximation process dependent on the rate that each individual parameter is taken to its limit. This subtlety is referred to as a problem of *distinguished limits.*

One of the core themes that underlies much of the study of exponential asymptotics is the generic divergence of singular asymptotic expansions. Although the typical mathematics education in analysis often begins from the investigation of convergent series, convergence is perhaps less useful in practice than we are led to believe. The *unreasonable effectiveness* of divergent series in providing accurate approximations has led to a degree of folklore[8]. We demonstrate this via an example.

Consider an approximation of the error function [Abramowitz and Stegun, 1983, Chap. 7]:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\, \mathrm{d}t, \tag{2.13}$$

where z is considered to be real. First, if the exponential e^{-t^2} is expanded about $t = 0$ and the integral is evaluated term-by-term, we obtain

$$\begin{aligned}\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \mathcal{O}(z^9)\right)\\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}.\end{aligned} \tag{2.14}$$

By the ratio test[9], the above series approximation is convergent for all values of z. Note that the power series near $z = 0$ is an asymptotic expansion for $z = \epsilon$ as $\epsilon \to 0$.

On the other hand, if the behaviour as $z \to \infty$ is first extracted from the integral, and the result integrated-by-parts, we obtain

$$\begin{aligned}\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}}\left(\int_0^\infty - \int_z^\infty\right) \mathrm{e}^{-t^2}\, \mathrm{d}t\\ &= 1 - \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1\cdot 3}{(2z^2)^2} - \frac{1\cdot 3\cdot 5}{(2z^2)^3} + \dots\right),\\ &= 1 - \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2z^2)^n}\end{aligned} \tag{2.15}$$

and a resultant series approximation that is divergent for all values of z. Examining fig. 2.4 of the error (exact value minus approximate value) as a function of the number of terms kept in each approximation, we see in fact that the convergent series converges much too slowly to be of practical use for most values of z. For instance, when $z = 5$, the convergent series contains an error of approximately 10^8 at a truncation of $N = 20$ terms. The divergent series, however, is accurate to 10^{15} with $N = 1$.

For a given value of z, the minimal error of the divergent series is obtained at the optimal truncation point. Heuristically, this is found at the value of N where adjacent terms are approximately equal in size, or in this case

$$\left|\frac{(2n-3)!!}{2z^2(2n-1)!!}\right| = \left|\frac{2n+1}{2z^2}\right| \sim 1.$$

When $z = 5$, the optimal truncation point is $N = \lceil 24.5 \rceil$.

[8] [Boyd, 1999, p. 9] has coined Carrier's Rule after George F. Carrier "Divergent series converge faster than convergent series because they don't have to cover".

[9] Note that the absolute ratio of series coefficients, $|a_{n+1}/a_n|$ yields $|z^2(2n+1)/[(2n+3)(n+1)]| \to 0$ for any fixed value of z and for $n \to \infty$

Note that the double factorial function is defined via $m!! = m(m-2)(m-4)\dots 1$.

"The idea that a function could be determined by a divergent asymptotic series was a foreign one to the nineteenth century mind. Borel, then an unknown young man, discovered that his summation method gave the "right" answer for many classical divergent series. He decided to make a pilgrimage to Stockholm to see Mittag-

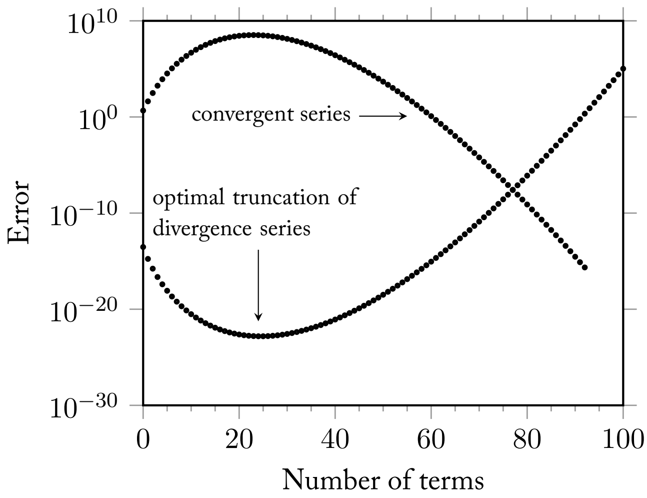

Figure 2.4: The vertical axis indicates the error of the approximation of $\operatorname{erf}(z)$ as a function of the number of terms included when $z = 5$. The top curve is the convergent series (2.14) and the bottom curve is the divergent series (2.15).

2.6 FURTHER REFERENCES

Many undergraduate courses will introduce perturbation theory through the analysis of algebraic equations, and Hinch [1991] develops the initial theory in this fashion. A number of our presentation of the basic definitions can also be found in Murray [2012] and Holmes [2012], the latter book providing an excellent introduction to the method of dominant balance. The book by White [2010] also provides a clear, succinct introduction to elementary asymptotic analysis.