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# 1

## 1.1 MOTIVATION

We provide a simple, yet deep, example of exponential asymptotics in the following classic example of a geometric model for crystal growth first proposed by Brower et al. [1983, 1984], and then subsequently studied by Kruskal and Segur [1991]. The problem is to consider the motion of the solid-liquid interface of a growing dendritic crystal where the dynamics are governed purely by the local geometry of the interface.

For the two-dimensional model of a crystal as shown in fig. 1.1, let $\phi$ be the angle of the normal of the interface to the $x$-axis, and depends on the arclength $s$ along the interface. We assume that $v_n$, the normal velocity of the interface depends on a function of the local curvature $\kappa = \frac{\mathrm{d}\phi}{\mathrm{d}s}$ and its derivatives.

Once non-dimensionalised, the most basic assumption is that

$$v_n = \kappa + \epsilon^2 \kappa_{ss}, \tag{1.1}$$

where the addition of the even differential in $\kappa$ stabilises the interface at short distances and plays the role of surface tension [Hakim, 1991].

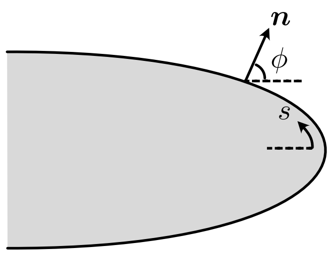


Figure 1.1: Geometry of the model for crystal growth. The arclength is such that $s \to -\infty$ on the lower side and $s \to +\infty$ on the upper side. The interface is assumed to move in the $x$ direction with velocity $v$.

We consider a further simplified problem where the crystal moves at a steady and constant unit velocity, where the normal velocity is $v_n = \cos\phi$. The problem is thus to determine the interface angle $\phi$ as a function of the arclength $s$. The interface is assumed to be governed by

$$\epsilon^2 \frac{\mathrm{d}^3\phi}{\mathrm{d}s^3} + \frac{\mathrm{d}\phi}{\mathrm{d}s} = \cos\phi, \tag{1.2a}$$

$$\phi \to \pm\frac{\pi}{2} \quad \text{as } s \to \pm\infty. \tag{1.2b}$$

With $\epsilon > 0$ small, we consider expanding the solution as a regular expansion:

$$\phi(s) = \phi_0(s) + \epsilon^2\phi_1(s) + \ldots \tag{1.3}$$

Noting that

$$\cos\phi = \cos\phi_0 - \epsilon^2\phi_1 \sin\phi_0 + \mathcal{O}(\epsilon^4), \tag{1.4}$$

we have from (1.2a), the leading-order problem:

$$\phi_0' = \cos\phi_0, \tag{1.5}$$

[1]See exercises.

We can verify[1] that the solution is given by

$$\phi_0(s) = -\frac{\pi}{2} + 2\tan^{-1}(\mathrm{e}^s), \tag{1.6}$$

which satisfies the necessary boundary condition of $\phi_0 \to -\pi/2$ as $s \to -\infty$. Note that the above solution was developed for a first-order differential equation, and yet also $\phi_0 \to \pi/2$ as $s \to \infty$, which may seem either fortuitous or suspicious.

The next order terms, from (1.2a) yields

$$\phi_1' + \phi_1 \sin\phi_0 = -\phi_0''', \tag{1.7}$$

and this can again be solved explicitly, yielding:

$$\phi_1(s) = -(2 + s - 2\tanh s + C)\,\mathrm{sech}\, s, \tag{1.8}$$

where the constant of integration, $C$, is left undetermined at this stage. Notice that $\phi_1 \to 0$ as $s \to \pm\infty$ and hence the perturbative solution at this stage generically satisfies the boundary condition regardless of the choice of $C$ (which fixes the origin). It can furthermore be shown [Kruskal and Segur, 1991] that a solution can be determined at every order that satisfies the boundary conditions.

However, examination of the numerical solutions shows a different story. In fig. 1.2, the differential equation (1.2a) is solved using a standard finite difference numerical scheme over the interval $-10 \leq s \leq 20$. The solution is initiated with approximate numerical boundary conditions of $\phi = 10^{-4}$, $\phi' = \phi'' = 0$ at the left endpoint, $s = -10$.

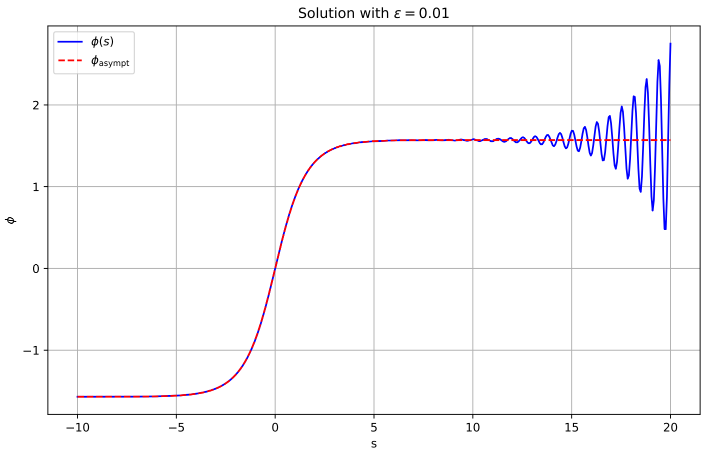


Figure 1.2: Numerical solution of (1.2a) with $\epsilon = 0.01$ with approximate boundary conditions on the left endpoint, $s = -10$.

We observe that regardless of the value of $\epsilon$, it seems that numerical solutions always exhibit oscillations in the far-field. Once measured, it can be verified that, at a fixed value of $s$, these oscillations are exponentially small as $\epsilon \to 0$. Therefore, there exists a contribution that is beyond-all-orders of the regular asymptotic expansion. Later, we will show that:

$$\phi \sim \Big[\phi_0(s) + \epsilon^2\phi_1(s) + \epsilon^4\phi_2(s) + \dots\Big] + \frac{A}{\epsilon^\gamma}\mathrm{e}^{-k/\epsilon}\sqrt{\cosh(s)}\cos(s/\epsilon + \psi), \quad (1.9)$$

with $k > 0$, and $A$, $\gamma$, and $\psi$ are constant.

There are several key points we want to highlight in this example.

**Divergence.** If we examine the leading-order solution, (1.6), we notice that it is smooth and infinitely differentiable along the real $s$-axis. However, since

$$\tan^{-1}(u) = \frac{\mathrm{i}}{2}\log\left(\frac{u+\mathrm{i}}{u-\mathrm{i}}\right), \quad (1.10)$$

the arctan function is singular wherever its argument is $\pm\mathrm{i}$. Thus the $\phi_0$ contains (logarithmic) singularities in the complex plane wherever $\mathrm{e}^s = \pm\mathrm{i}$ or firstly at $s = \pm\pi\mathrm{i}/2$ and the subsequently spaced $2\pi$ up and down the imaginary axis. Similarly at the next order $\phi_1$ contains singularities such as $\phi_1 \sim c(s - \pi\mathrm{i}/2)^{-2}$.

Considering the procedure in deriving the additional terms of the differential equation, we remark that it involves terms such as

$$\phi_{n-1}''' + \phi_n' = \dots \quad (1.11)$$

therefore, obtaining the $n$th term of the approximation typically involves differentiation of the previous order twice. At next order, we might expect $\phi_2$ to contain a factor such as $2 \cdot (s - \pi\mathrm{i}/2)^{-4}$. Hence each order beings additional multiplicative factors of ever-increasing number, and a greater power in the denominator. We might thus conjecture that, no matter what value of $s$ is chosen, $|\epsilon^n \phi_n| \to \infty$ as $n \to \infty$—at any chosen fixed $\epsilon$.

This illustrates a fundamental fact of almost all singularly perturbed differential equations that we shall study: their regular asymptotic expansions diverge. The question, then, of the sensibility of using a divergent expansion in order to capture a (presumably well-defined) solution of a physical problem is raised.

**Beyond-all-orders.** Above, we have only shown the leading contribution of the beyond-all-orders contribution. Indeed, there are terms

$$\sim \frac{\mathrm{e}^{-k/\epsilon}}{\epsilon^\gamma}\Big[A_0(s) + \epsilon^2 A_1(s) + \epsilon^4 A_2(s) + \dots\Big]. \quad (1.12)$$

Moreover, because of nonlinearity in the ODE, we know that there are likely terms of order $\mathrm{e}^{-2k/\epsilon}$, $\mathrm{e}^{-3k/\epsilon}$, ... There indeed seems to be quite a fearsome structure (or tower) of asymptotic terms beyond-all-orders of the routine regular expansion.

**Does 'exponentially-small' mean 'negligible'?** Here we have a clear example where exponential smallness does not mean negligible. In this case, the situation is worse: the $\sqrt{\cosh(s)}$ grows to be exponentially dominant as $s \to \infty$, strongly invalidating the necessary boundary conditions. However, even if not for this latter factor, the non-decaying nature of the exponentially-small ripples with factor $e^{-k/\epsilon}$ would invalidate the boundary conditions. This is a clear illustration that the exponential asymptotics is by no means negligible, and in fact, it is central to the existence (or non-existence) of solutions in this case.

## 1.2 EXERCISES

1. Solve for the leading-order crystal-growth solution (1.6).

   Note the antiderivative of $\sec(u) = 1/\cos(u)$ is $\log|\sec(u) + \tan(u)| + C$. It is useful to also use the trigonometric identities:

   $$\sin\theta = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad \cos\theta = \frac{x}{\sqrt{1+x^2}},$$

   where $\theta = \tan^{-1}(x) = \tan^{-1}(e^s)$. Also $\arctan u = \frac{i}{2}\log\left(\frac{u+i}{u-i}\right)$.
2. Write and implement your own finite difference solver in order to study numerical solutions of (1.2). Investigate the effect of the domain size, initial boundary conditions, and $\epsilon$ on the solutions. Is it possible to measure the amplitude of the downstream waves in order to intuit the necessary asymptotic form as $\epsilon \to 0$?
3. By linearising the ODE (1.2a) as $s \to \pm\infty$ about $\pm\pi/2$, develop the linear terms, i.e. $\phi = \pm\pi/2 + f(s)$ where $f$ is small. What does the possible solutions for $f$ inform you about the required boundary conditions?