2003-2009
Summary of Research Work

Philippe H. Trinh
Oxford Centre for Industrial and Applied Mathematics
Mathematical Institute, University of Oxford
E-Mail: trinh@maths.ox.ac.uk
Web: http://www.ptrinh.com

Reverse Chronological Summary

2007-2010	The Existence and Non-Existence of Waveless Ships
2007-2010	Exponential Asymptotics and Gravity-Capillary Waves
2006-2007	Resonant Solutions of a Broad Class of Forced KdV Equations
2005	Steady Solutions of a Coupled KdV System
2004-2005	Genomic Features in the Breakpoint Regions Between Syntenic Blocks
2003-2004	Chromosomal Breakpoint Re-Use in Genome Sequence Rearrangement

Research Interests

A vast portion of current research on scientific phenomena lies on the boundaries between different physical or mathematical theories—classical and quantum, rays and waves, linear and nonlinear, laminar and turbulent, and so on. Mathematically, these boundaries are often posed as questions in physical (often singular) asymptotics—the study of limits. This is my area of interest.

In the broadest sense, my goal as an applied mathematician is to develop analytical and numerical methodologies for the study of ordinary and partial differential equations. In particular, I have taken a recent interest in problems in fluid mechanics which involve the *breakdown* of traditional asymptotic analysis, subsequently requiring the use of asymptotics *beyond-all-orders*.

For example, in the study of nonlinear free-surface flows, the relative intractability of the surface equations generally requires consideration of the solution as a perturbative expansion,

$$S = \sum_{n=0}^{\infty} \epsilon^n S_n,$$

which depends on the limit of some small parameter, ϵ tending to zero. But the limit $\epsilon \to 0$ is often singular and difficulties arise; the series S is *divergent* and consequently by the Stokes Phenomenon, exponentially small terms can suddenly appear and disappear in the analytic continuation of the problem. The body of methods that are developed for their study is called *exponential asymptotics* (or *asymptotics beyond-all-orders*), and applications of these special techniques have been found in diverse problems, ranging from models of crystal growth and viscous fingering [1], to the onset of turbulence [2], the rupturing of thin films [3], and the study of free-surface flows [4].

Career Path

My introduction to the world of science, research, and academia occurred at a young age. In high school (2000-2004), I was given an amazing opportunity to work with David Sankoff at the University of Ottawa on the problems in mathematical genomics; here, the work involved the statistical analysis of data from various mammalian genomes, as well as the modeling of genome sequence rearrangement. However, during my undergraduate and Master's degree, I moved on to work with Dave Amundsen at Carleton University on asymptotic techniques applied to resonant interactions in Korteweg-de Vries equations (2004-2007). For my Ph.D (2007-2010) at the Oxford Centre for Industrial and Applied Mathematics (OCIAM), I have been studying the application of exponential asymptotics to the study of free-surface flows.

1 The Existence and Non-Existence of Waveless Ships

Date:	2007-Present (Ph.D.)
Supervised by:	Prof. Jonathan Chapman (Oxford) Prof. Jean-Marc Vanden-Broeck (UCL)
Funded by:	Clarendon Fellowship (Oxford) NSERC PGS-D (Canada)

Summary of Work

When an ideal fluid flows past a surface-piercing object or over an obstruction, waves are sometimes produced upstream or downstream of the disturbance. But in the low-speed limit, the traditional asymptotic series in powers of the Froude number fails to capture this phenomenon—this is the so-called *Low-Speed Paradox* first mentioned by Ogilvie [5]. It is now known that at low Froude numbers, the waves are in fact exponentially small and thus *beyond-all-orders* of regular asymptotics; their formation is a consequence of the divergence of the asymptotic series and the associated Stokes Phenomenon.

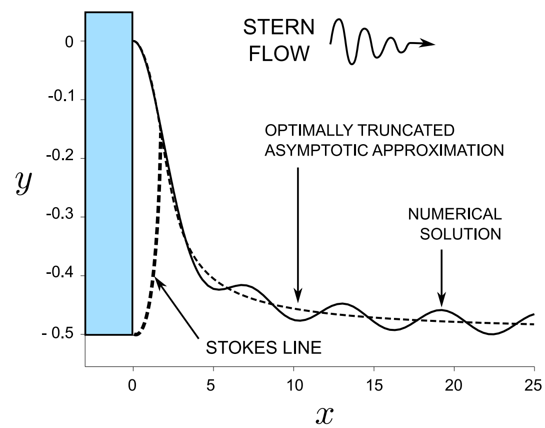

Figure 1: The *Low-Speed Paradox* addresses the fact that that asymptotic expansions in powers of the Froude number fail to capture waves present on the free surface in potential flow problems. In this figure, the asymptotic (waveless) expansion of stern flow is compared to the numerical solution of the problem. The key idea is that a Stokes line emerges from the corner-singularity, across which an exponentially small wave turns on.

This underlying subtlety has been painfully problematic in regards to previous asymptotic and numerical treatments of the nonlinear ship-wave problem. In [6], Dagan and Tulin showed that the analysis near a three-dimensional ship can be reduced to studying the two-dimensional ideal flow problem where the ship is modeled as a semi-infinite body with constant draft. This fully nonlinear free-surface problem was first computed by Vanden-Broeck and Tuck [7], and on the basis of numerical evidence, they conjectured that ship hulls with a single front face will always generate waves (see Figure 1).

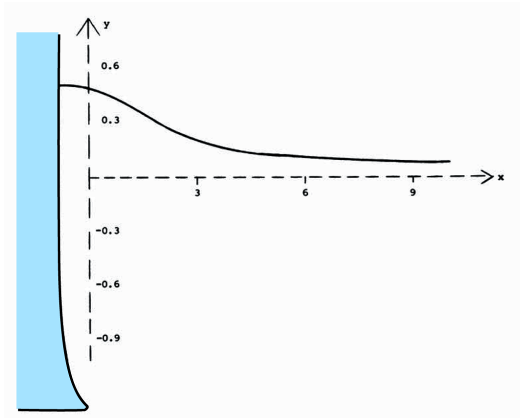

FIGURE 2: The seemingly waveless bulbous ship (their Figure 6) numerically discovered by Tuck and Vanden-Broeck [8] was later refuted by Farrow and Tuck [9]. Do there exist ship hulls which are effectively waveless?

Moreover, the earlier experimental work of Baba [10] had indicated that a bulbous bow can eliminate, or at least reduce the splash at the bow of a ship[1]. This prompted the discovery of seemingly waveless ships with bulbous profiles, first by Tuck and Vanden-Broeck [8] and later confirmed by Madurasinghe [11]—but again, only *numerically* so (Figure 2). Unfortunately, these results were refuted by the more comprehensive numerical study of Farrow and Tuck [9]; there, they wrote,

> The free surface would at first sight appear to be waveless, but on closer examination of the numerical data, there are very small waves present and they have a steepness of 1.5×10^{-3}.

Clearly, these are questions which cannot be easily answered using simple numerics. Indeed, in *Reminiscences and Reflections: Ship Waves, 1950-2000* [12], Tulin mentions two pressing open questions:

> The fundamental questions of whether such rising potential free-surface flows before bluff bodies exist [...] still remain open,

and

> Is it demonstrable [...] that continuous solutions will not exist in the limit of vanishing speed? Does this have anything to do with the inability of Tuck and his colleagues [...] to find a continuous solution in the two-dimensional bow wave case? Do nonbreaking flows exist at all for surface-piercing ship forms of arbitrary form and thickness, at any speed?

[1] In the potential flow problem, a waveless solution past the stern (rear) of a ship is equivalent to a splashless solution at the bow (front) of a ship.

However, with the recent development of techniques in exponential asymptotics (see for example [4, 13]), many of these issues can be resolved.

Our key result is the demonstration that the formation of waves near a ship is a necessary consequence of singularities in the ship's geometry or its analytic continuation, such as those corresponding to sharp corners. Afterwards, the use of complex-plane asymptotics, optimal truncation, and Stokes line smoothing can be applied to detect the Stokes lines emerging from each singularity, and then to derive the form of the exponential switched on as the line is crossed.

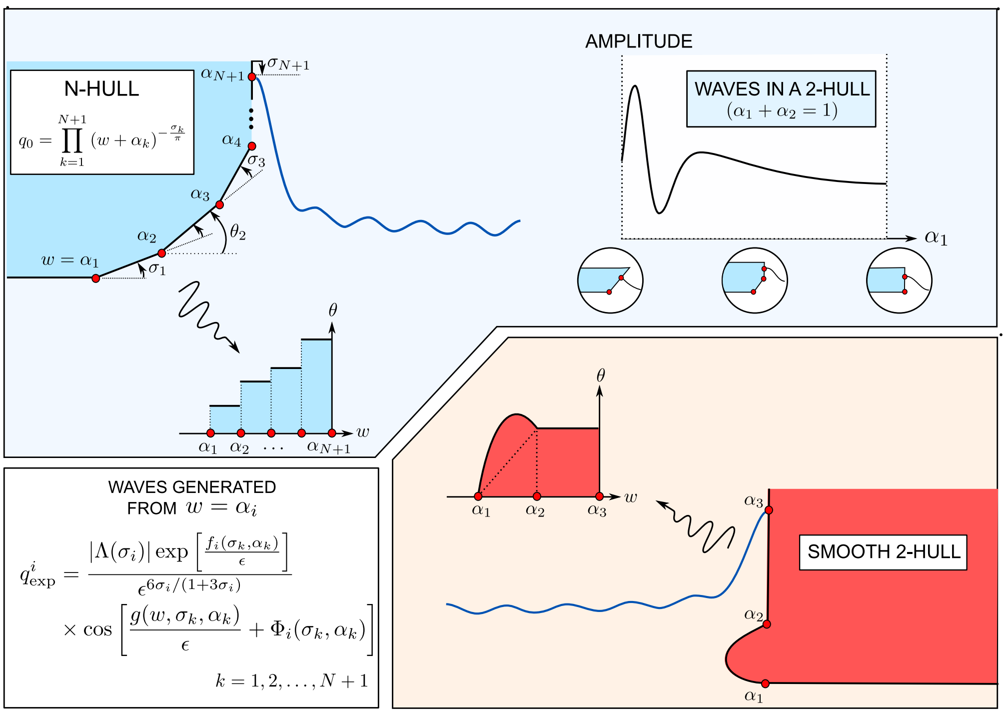

FIGURE 3: Exponential asymptotics can be applied to study the waves behind an *N-Hull*, a linear-piecewise ship with N corners (top left). If the contributing singularities are sufficiently widely spaced, each corner produces a wave with a given analytical form (bottom left); if not, then 'inner regions' with multiple singularities must be studied (top right). Research on the case of smoothed hulls is ongoing (bottom right).

The theory can also be extended to the study of ships that are not only piecewise linear, but also piecewise entire (in their analytic continuation); this includes the class of bulbous hulls previously studied by the aforementioned authors (see Figure 3).

Finally, the analysis has been applied in order to prove that certain ship profiles will or will not produce a wake in the low-speed limit. For example, ships with an odd-number of corners can be shown to always generate a non-zero wake (in particular, this confirms the conjecture by Vanden-Broeck and Tuck [7] for the rectangular stern). Analytical criteria can also be derived for the existence of waveless ship forms (see Figure 4).

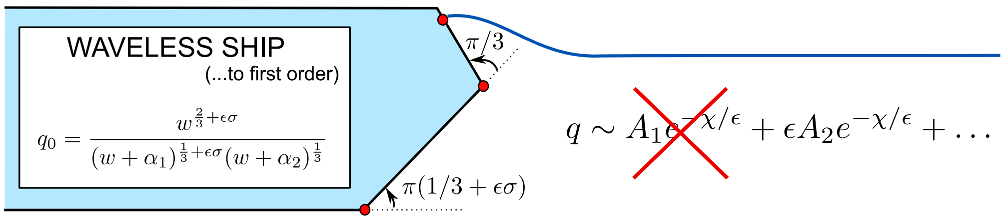

FIGURE 4: Our asymptotic analysis can be used to produce *almost* waveless ships. In this figure, we illustrate a ship with divergent corner-angles of approximately $\pi/3$ that can be made waveless to first order in ϵ for the correct choice of α_1, α_2, and σ. It has a bulbous shape, and the wave contributions from each distinct corner produces perfect phase cancellation (to first order).

Publications in Preparation and Conference Proceedings

P. Trinh and S.J. Chapman (2010) "The Existence and Non-Existence of Waveless Ships," J. Fluid Mech. (In preparation)

P. Trinh and S.J. Chapman (2010) "Exponential Asymptotics and Bulbous Ships," Physics of Fluids (In preparation)

P. Trinh, S.J. Chapman and J.-M. Vanden-Broeck (2008) "Exponential Asymptotics and Stern Waves," Proc. 5th IMACS Int. Conf. Evol. Equat. Nonl. Waves, Athens, Georgia.

Talks

P. Trinh (2009) "Do Waveless Ships Exist?", a talk at the DFD09 Meeting of The American Physical Society in Minneapolis, Minnesota

P. Trinh (2009) "The Existence and Non-Existence of Waveless Ships?", a talk at the 14$^{\text{th}}$ Cambridge/Oxford Applied Mathematics Meeting in Cambridge, UK.

P. Trinh (2009) "How can exponential asymptotics be used to find hidden surface waves in low-speed flows?", a talk at the British Applied Mathematics Colloquia (BAMC) in Nottingham, UK.

P. Trinh (2009) "Asymptotics beyond-all-orders and the Devil: Mathematical, physical, historical, and philosophical significance in 15 minutes or less", a talk at the Balliol Jowett Scholarship competition in Oxford, UK.

P. Trinh (2008) "Exponential asymptotics and stern waves", a talk at the 5$^{\text{th}}$ IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena in Athens, Georgia.

2 Exponential Asymptotics and Gravity-Capillary Waves

Date: 2007-Present (Ph.D.)
Funded by: NSERC PGS-D (Canada) and Clarendon Fellowship (Oxford)
Supervised by: Prof. Jonathan Chapman (Oxford)

Summary of Work

It is well known that the study of water waves with both gravity and surface-tension present contains numerous difficulties not found when only one effect is included. For example, in the classical fishing rod experiment of Rayleigh [14], linearized theory predicts several regimes of interest for different values of the Froude and Bond numbers. However, the nonlinear structure of the problem is much more complex and many analytical and numerical studies have sought to unravel the full spectrum of solutions (see for example [15, 16]).

Much less is known on the topic of gravity-capillary flows past obstructions which cannot be considered small. For example, we could ask what is the effect of placing an order-one trapezoidal object on the bottom of a stream and how it differs from using a semi-circular object or a step[2]. The standard technique is to linearise for small obstructions and thus the geometry of the obstruction is 'lost'. Instead, we propose that the low-Froude and low-Bond approximation can be used to elucidate analytical details of these problems—under this simplification, the geometry of the obstruction is preserved at the expense of dealing with asymptotic divergence. Hence this is a singular perturbation problem, Stokes lines can be expected, and techniques in exponential asymptotics must be used to observe the switching-on of waves (as illustrated in Figure 5).

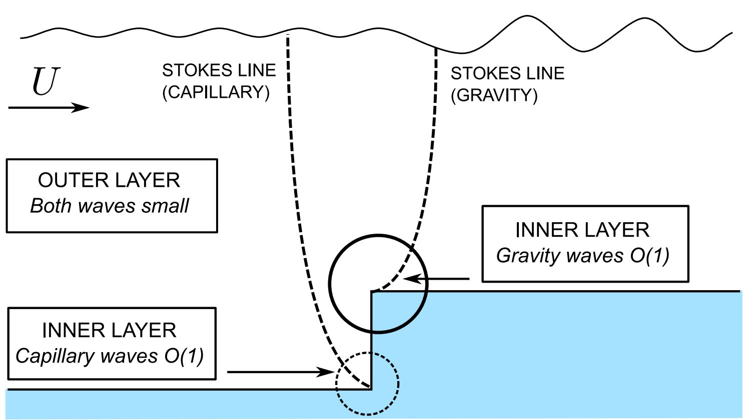

Figure 5: In the case of a rectangular step, there exist Stokes lines originating from the corner and stagnation points. There also exist additional turning points (not shown) corresponding to when the two exponentially-small waves are in phase; these may lie either on the bottom bed or on the free surface. As the Froude and Bond numbers are varied, different solutions are possible.

[2] To the author's knowledge, this gravity-capillary problem with nonlinear geometry has only been numerically studied by Forbes [17], and Grandison and Vanden-Broeck [18]. While both studies attempt to solve the full nonlinear water-wave equations, only the solutions for relatively small bottom obstructions were found.

Here is an example of the novelty in this approach. When ideal fluid flow is considered over a step, a linearisation of the free surface and the step height produces an approximation which replaces the step by a point source. As the Froude and Bond numbers tend to zero, there exist Stokes lines originating from the point and intersecting the free surface. If there are two *complex* wavenumbers to the linearised problem, then there are two symmetrical Stokes lines; if there are two *real* wavenumbers, then the two Stokes lines coalesce into one. Further analysis then tells us that in the former case, the free surface is of solitary type, while in the latter, there are capillary waves upstream and gravity waves downstream.

Now instead, if we preserve the nonlinearity of the step and take the limit as the Froude and Bond numbers tend to zero, the Stokes line structure changes dramatically. Now, there are Stokes lines emanating from the corner and stagnation point of the step (Fig. 5), as well as turning points (which may also produce Stokes lines) lying along either the boundary or the free surface. As the balance of the Froude and Bond numbers change, the Stokes line structure also changes, delineating regimes with different solutions. Even more subtle is what happens if the geometry of the obstruction is changed, say to an angled step or a wedge. Now, a single singularity may in fact generate *multiple* Stokes lines.

There is thus a rich variety of possibilities of changing the Stokes-line structure and thus the range of possible solutions, depending on the choice of the bottom topography as well as the crucial balance between Froude and Bond numbers—possibilities that have in the past, gone undetected by regular linear or weakly nonlinear theories.

Our research seeks (1) to classify the different solutions possible in the low-Froude, low-Bond limit, (2) to develop the exponential asymptotics necessary to derive the exponentially small surface waves in each regime, and (3) to solve the full gravity-capillary problem numerically[3].

Publications in Preparation and Conference Proceedings

P. Trinh and S.J. Chapman (2010) "Exponential Asymptotics and Gravity-Capillary Waves," J. Fluid Mech. (In preparation)

P. Trinh (2009) "Preserving Nonlinear Geometries in Free-Surface Fows," CISM Courses and Lectures. (In preparation and conjunction with Asymptotic Methods in Fluid Mechanics course in Udine, Italy)

P. Trinh and S.J. Chapman (2009) "Exponential asymptotics and gravity-capillary waves" Proc. 9th Int. Conf. Math. Num. Asp. Waves (Waves 2009), To Appear.

Talks

P. Trinh (2009) "Preserving Nonlinear Geometries in Free-Surface Flows", a talk at the CISM Asymptotic Methods in Fluid Mechanics course in Udine, Italy.

[3]In the subcritical case where capillary waves exist upstream and gravity waves downstream, this is nontrivial problem due to the requirement of imposing an unknown upstream radiation condition. Nobody has yet proposed a truly robust method for numerically solving the full problem—see [18].

3 Unifying the Resonant Solutions for a Broad Class of Forced Korteweg-de Vries Equations

Date:	Summer 2006 (NSERC) Fall/Winter 2006-2007 (Master of Science)
Supervised by:	Prof. Dave Amundsen (Carleton)
Funded by:	NSERC Undergraduate Research Award Departmental Graduate Scholarship NSERC PGS-M

Summary of Work

The Korteweg-de Vries (KdV) equation was originally motivated by the problem of modeling nonlinear waves in shallow water. It has been extensively studied in the past and the mathematical theory behind the equation is rich and interesting. However in recent years, generalizations and variations of the original KdV have been proposed in conjunction with other physical systems. In particular, these include

$$u_t - \gamma u_{xxx} + \Delta u_x + \alpha u u_x - \mu u_{xx} = f(x) \quad (1)$$

$$u_t - \gamma u_{xxx} + \Delta u_x + \beta u^2 u_x - \mu u_{xx} = f(x) \quad (2)$$

$$u_t - \gamma u_{xxx} + \Delta u_x + \alpha u u_x + \beta u^2 u_x - \mu u_{xx} = f(x) \quad (3)$$

where $\alpha \neq 0$, $\beta \neq 0$, γ, Δ, and μ are constants. Equations (1), (2), and (3) represent a more general periodically forced and damped KdV, modified KdV (mKdV), and extended KdV (eKdV) equation, respectively. Each equation has been associated with a wide range of physical applications, from shallow water in a tank subjected to forcing at one end [19], to transcritical flow of a stratified fluid [20], and resonant flow over a topography [21].

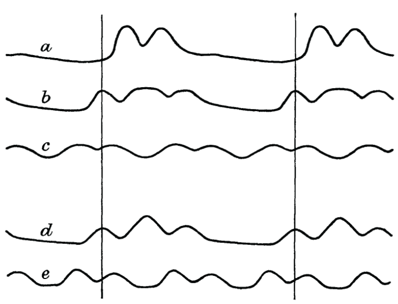

Figure 6: Chester and Bones [19] experimentally studied the problem of shallow water in a tank, forced near resonance. This is a graph (their Figure 3) of the surface elevation in time at different longitudinal points in the tank (a)-(e). The solutions consist of rapidly varying (dispersive) boundary layers, superimposed on a slowly varying profile.

In problems where both nonlinear and forcing effects are strong—such as in the water-wave experiments of Chester and Bones [19] in Figure 6—it is known that a rich array of steady solutions emerge. And although analytic studies in this regime have been performed, for instance in [22, 23, 24, 25], they still remain limited in their scope. Recently however, Amundsen, Cox, and Mortell [26] developed a general framework based on singular perturbation and asymptotic matching in the context of the forced KdV. Our work has been to extend these techniques to the case of the mKdV and eKdV equations, thus providing a more global perspective of the similarities and differences between the various KdV-type equations.

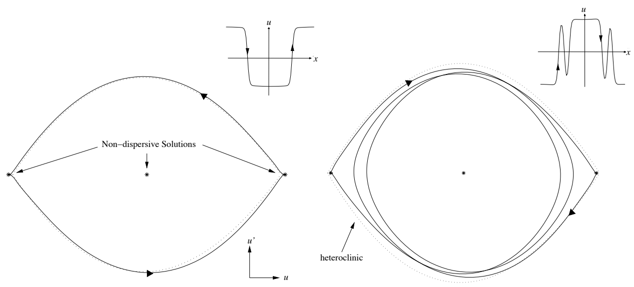

Figure 7: Typical single (left) and multiple (right) orbit cases for the mKdV equation near resonance. In the case of the left, only a single jump occurs between non-dispersive media and the trajectory follows closely with the heteroclinic orbit. In contrast, the right figure depicts a trajectory that orbits the nodal solution one and a half times before matching with the flanking saddle.

Indeed, our work has shown that the connection of the various KdV-type equations can be understood in terms of the behaviour of their respective solutions in the phase plane. For example, in the KdV, boundary-layer transitions occur from and to a single saddle point, while for the mKdV, the solutions transition *between* distinct saddles (Fig. 7). The eKdV, however, contains both KdV-like and mKdV-like transitions, depending on the particular balance of parameters. The resonant response of these equations can then be explored using AUTO and the solutions, as well as the critical transitions between the various KdV-type equations can be approximated using asymptotic analysis with multiple-scales (Fig. 8).

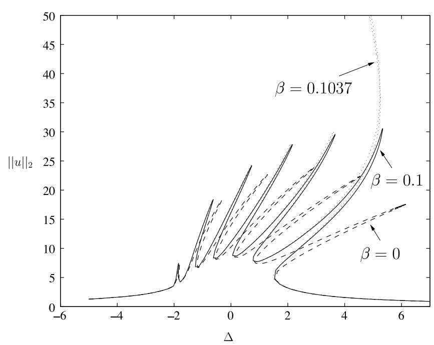

Figure 8: Resonant response for the eKdV equation with dispersion $\gamma = .005$, damping $\mu = .0015$, forcing $f(x) = \cos \pi x$, quadratic nonlinearity $\alpha = 2$, as the cubic nonlinearity β is increased. Near the value $\beta \approx 0.1037$, a homoclinic orbit turns into a heteroclinic orbit.

Publications, Conference Proceedings, and Talks

P. Trinh and D. Amundsen (2009) "Unifying Steady-State Resonant Solutions of a Broad Class of KdV-Type Equations" *J. Compu. Appl. Math*, In Press.

P. Trinh (2008) "Unifying resonant solutions of a broad class of KdV equations", a talk at the Junior Applied Mathematics Seminars (JAMS) in Oxford, UK.

P. Trinh and D. Amundsen (2007) "Resonant solutions of the periodically forced eKdVB equation" Proc. 8th Int. Conf. Math. Num. Asp. Waves (Waves 2007), University of Reading, 120-122.

4 Steady Solutions of a Coupled Korteweg-de Vries System

Date: Summer 2005 (NSERC)
Supervised by: Prof. Dave Amundsen (Carleton)
Funded by: NSERC Undergraduate Research Award

Summary of Work

In [27], Hirota and Satsuma propose a system of two coupled KdV equations,

$$\begin{aligned} u_t + a(u_{xxx} + 6uu_x) &= 2bvv_x \\ v_t + v_{xxx} + 3uv_x &= 0 \end{aligned}$$

which can be used to describe interactions of two long waves with different dispersion relations. This system has the property that the existence of each long wave affects the behaviour of the other. Following their work, there arose considerable interest in coupled KdV systems using different coupling parameters or involving more than two coupled equations. Particular applications have been found in many fields, such as nonlinear optics, fluid dynamics, and so forth.

The model which was considered in this work was a more general system of coupled KdV equations,

$$\begin{aligned} u_t + \gamma_u u_{xxx} + \mu_u u_{xx} + \Delta_u u_x + \eta uuu_x &= \kappa_u v_x \\ v_t + \gamma_v v_{xxx} + \mu_v v_{xx} + \Delta_v v_x + \eta vvv_x &= \kappa_v u_x \end{aligned}$$

where the γ, μ, Δ, η, and κ are constants and control the amount of dispersion, damping, detuning from resonance, non-linear effects, and coupling strength, respectively.

Based on the underlying understanding of the forced KdV equation, we were able to see some similarities with the uncoupled variant, but the presence of coupling feedback introduced a rich array of novel and non-trivial characteristics, many of which are now the focus of ongoing further research.

Internal Reports

P. Trinh (2005) "Steady Solutions of a Coupled Korteweg-de Vries Equation," Internal report submitted to NSERC USRA 2005.

5 Genomic Features in the Breakpoint Regions Between Syntenic Blocks

Date: 2004
Supervised by: Prof. David Sankoff (University of Ottawa)

Summary of Work

We study the largely unaligned regions between the syntenic blocks conserved in humans and mice, based on data extracted from the UCSC genome browser. These regions contain evolutionary breakpoints caused by inversion, translocation and other processes.

Results indicate that the number, size, and distribution of small aligned fragments in the breakpoint regions depend on the origin of the neighbouring blocks and the other blocks on the same chromosome.

In addition to the purely statistical analysis, explanations are suggested for the limited amount of genomic alignment in the neighbourhood of breakpoints. In particular, the distribution of fragments in the breakpoints is accounted for partially by artifacts due to alignment protocols and partially by mutational process operative only after the rearrangement process.

Publications and Conference Proceedings

P. Trinh, A. McLysaght, and D. Sankoff (2004). "Genomic features in the breakpoint regions between syntenic blocks", *Bioinformatics*, **20(1)** 318-325.

6 Chromosomal Breakpoint Re-use in Genome Sequence Rearrangement

Date: 2003-2004
Supervised by: Prof. David Sankoff (University of Ottawa)

Summary of Work

In comparing various genomes, the appearance of both large and small scale rearrangements can be seen, the results of millions of years of evolution. However, the lack of complete data in numerous small regions is particularly problematic when applying gene-order rearrangement algorithms, which allow for a study of the historical evolution of the genomes.

Pevzner and Tesler developed a method which focuses on major evolutionary events by neglecting small blocks under a threshold length. In doing so, they concluded that certain areas of the human-mouse genomes were more susceptible to small-scale rearrangements—so-called fragile breakpoints.

However, the use of large "sanitized" blocks and the neglect of short blocks may blur important parts of the historical derivation of the genomes. To study this possibility, we apply a variety of analytic and simulation methods to show that the fragile nature of the genomes is highly dependent upon the threshold size and the particular parameters of the rearrangement process.

Publications and Conference Proceedings

D. Sankoff and P. Trinh (2005). "Chromosomal breakpoint re-use in genome sequence rearrangement" *J. Comput. Biol.*, **12** 812-821.

D. Sankoff and P. Trinh (2004). "Chromosomal breakpoint re-use in genome sequence rearrangement" *Proc. RECOMB '04.* New York: ACM Press, 30-35.

References

[1] Kruskal, M. and Segur, H. *Stud. Appl. Math.* **85**, 129–181 (1991).
[2] Chapman, S. *J. Fluid Mech.* **451**, 35–97 (2002).
[3] Witelski, T. and Bernoff, A. *Physica D* **147**, 155–176 (2000).
[4] Chapman, S. and Vanden-Broeck, J.-M. *SIAM J. Appl. Math.* **62 (6)**, 1872–1898 (2002).
[5] Ogilvie, T. Technical report, Michigan University, Ann Arbor, (1968).
[6] Dagan, G. and Tulin, M. *J. Fluid Mech.* **51 (3)**, 529–543 (1972).
[7] Vanden-Broeck, J.-M. and Tuck, E. In *2nd Internatinal Conference on Numerical Ship Hydrodynamics (Berkeley, Ca)*, (1977).
[8] Tuck, E. and Vanden-Broeck, J.-M. In *Proc. 15th Symp. Naval Hydrodynamics*, (1984).
[9] Farrow, D. and Tuck, E. *J. Austral. Math. Soc. Ser. B* **36**, 424–437 (1995).
[10] Baba, E. In *Proc. Int. Seminar on Wave Resistance, Tokyo*, 75–92, (1976).
[11] Madurasinghe, M. *J. Ship. Res.* **32**(3), 194–202 (1988).
[12] Tulin, M. *J. Ship Res.* **49**(4), 238–246 (2005).
[13] Chapman, S. and Vanden-Broeck, J.-M. *J. Fluid Mech.* **567**, 299–326 (2006).
[14] Rayleigh, L. *Proc. Lond. Math. Soc.* **15**, 69–78 (1883).
[15] Vanden-Broeck, J.-M. and Dias, F. *J. Fluid Mech.* **240**, 549–557 (1992).
[16] Dias, F., Menasce, D., and Vanden-Broeck, J.-M. *Eur. J. Mech. B/Fluids* **15 (1)**, 17–36 (1996).
[17] Forbes, L. *J. Fluid Mech.* **127**, 283–297 (1983).
[18] Grandison, S. and Vanden-Broeck, J.-M. *J. Eng. Math.* **54**, 89–97 (2006).
[19] Chester, W. and Bones, J. *Proc. R. Soc. Lond. A* **A306**, 23–39 (1968).
[20] Smyth, N. *J. Fluid Mech.* **192**, 287–312 (1988).
[21] Marchant, T. and Smyth, N. *J. Fluid Mech.* **221**, 263–287 (1990).
[22] Cox, E. and Mortell, M. *Z. Angew. Math. Phys.* **34**, 845–866 (1983).
[23] Cox, E. and Mortell, M. *J. Fluid Mech.* **162**, 99–116 (1986).
[24] Ockendon, J. and Ockendon, H. *J. Fluid Mech.* **59**, 397–413 (1973).
[25] Ockendon, J., Ockendon, H., and Johnson, A. *J. Fluid Mech.* **167**, 465–479 (1986).
[26] Amundsen, D., Cox, E., and Mortell, M. *Z. Angew. Math. Phys.* **58**(6), 1008–1034 (2007).
[27] Hirota, R. and Satsuma, J. *Phys. Lett. A* **89**, 168–171 (1981).